

Consumer Credit and Aggregate Demand*

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Abstract

Recent evidence suggests shocks to household balance sheets contributed to employment losses during the 2007 - 2009 recession in the U.S. Motivated by these findings, I consider a model in which credit provides liquidity to consumers. If available credit is sufficient to finance desired consumption, the model is identical to a standard growth model. However, when there is insufficient credit, consumer demand falls and output is inefficiently low. Credit constraints are endogenous and depend on the return to production. A demand shock can trigger an amplification process, where lower demand reduces the return to production, tightening available credit and reducing demand further. In a crisis state, there is equilibrium underconsumption and overproduction. In normal times, there is equilibrium oversmoothing: oversaving by wealthy households and overborrowing by poor households. There is a clear role for policy, as government spending stimulates demand and relaxes consumer borrowing constraints.

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1 Introduction

Several empirical studies have documented that the severity of the 2007 - 2009 U.S. recession was greater in counties with higher levels of household debt before the crisis.¹ These facts suggest that a fall in consumer demand arising from shocks to household balance sheets was a driving force behind the recession.

To explore this channel, I develop a model in which consumer credit frictions amplify fluctuations in demand. This amplification works in two stages. Households are subject to borrowing constraints set by banks at a level that prevents default. When borrowing constraints bind, demand falls, decreasing the return to production. This decreases output, and also makes defaulting less costly for households. This prompts banks to tighten lending standards, reducing demand further, and the cycle repeats.

When no borrowing constraints bind, the model nests a standard growth model. When borrowing constraints bind, the model can produce substantial amplification in response to demand shocks.² Comparing equilibrium decisions to the choices of a constrained planner, I find that the equilibrium exhibits underconsumption and overproduction when constraints are binding. Intuitively, higher demand and lower supply of final goods both drive up the return to production and relax borrowing constraints. Likewise, optimal government policy when constraints are binding calls for higher government spending to raise demand and relax borrowing constraints.

I build on standard models of heterogeneous agents with borrowing constraints, such as [Aiyagari \(1994\)](#) and more recently [Guerrieri and Lorenzoni \(2011\)](#). I add two ingredients to these models. First, I assume that agents receive the income from current production only after they make their expenditures for the period. Thus households cannot use current income to purchase consumption directly, but must draw on their savings or borrow against future income. Second, I derive credit constraints endogenously from the repayment problem of households, rather than setting them exogenously. Thus credit constraints depend on the aggregate state of the economy, generating amplification.

The delayed income assumption is motivated by the structure of production in modern economies. Due to the division of labor, agents rarely consume their own output, and there are often substantial lags between production, sale, and receipt of payment. Income to entrepreneurs from their ventures and rents to owners of capital are typically received well after the provision of these factors of production. Likewise, employees typically receive their paychecks at the end of a pay period.

¹See in particular [Mian and Sufi \(2010\)](#)

²Examples of demand shocks include fluctuations in credit supply, government spending, and investment demand due to shifts in expectations.

This assumption has two consequences. First, since income from current production cannot be used to finance current consumption directly, households must borrow to finance any consumption in excess of savings, even if they have positive net worth. Second, since households cannot increase available funds by working more, binding constraints do not increase production. These differ from standard models of consumer credit constraints, in which borrowing constraints only apply to a subset of households with negative net worth, and constrained households increase their production. The result is a powerful channel running from consumer credit to aggregate output.

The second component of the amplification process requires that borrowing constraints endogenously tighten following a fall in demand. In reality, credit conditions worsen during recessions due to high unemployment, falling value of collateral, and reduced profitability of firms. In my baseline model, borrowing constraints tighten following a fall in demand because this decreases the price of current goods relative to future goods, reducing the effective return to production. This tightens borrowing constraints because the penalty for default is seizure of assets, which are more valuable when the relative price of current goods is greater.

The price of current goods relative to future goods, which is equivalent to the real interest rate, is a key variable, as it determines both the supply of goods and the level of borrowing constraints. When constraints are binding, any decision that affects this key price will affect the consumption of constrained households. Since this effect is not considered by the agent making the decision, this represents a *pecuniary externality*. Relative to the choices of a constrained planner, households choose too little consumption and too much production when constraints are binding.

Optimal government policy should act to raise the real interest rate when constraints are binding. This can be done by increasing government spending to raise demand. The effect of tax policy depends crucially on the timing and nature of the tax. Any transfer to constrained households that arrives before they make their purchases will relax the constraint directly, yielding a large return. However, if the transfer is received after expenditures are made, they will have no effect. Moreover, marginal tax rates should be *increased* in order to reduce output supply, which will raise the real interest rate and relax the constraint. In fact, if constraints are binding for the majority of households, a reduction in marginal tax rates may actually *decrease* output. This is similar to the *Paradox of Toil* seen in New Keynesian models at the zero lower bound.³

The existence of this mechanism has further consequences for the distribution of wealth in society. If the distribution of initial assets is more unequal, then a smaller de-

³See Eggertsson (2010), and the discussion of these models in the Literature section below.

mand shock will be necessary to cause borrowing constraints to bind for some households. Since households do not consider the effect of their saving decisions on borrowing constraints during future crises, poor households will save too little and rich households will save too much relative to the choices of a constrained planner. Consequently, policies that reduce the spread in the distribution of assets will reduce the vulnerability of the economy to future crises.

In the rest of this paper, I develop a tractable model framework to explore this mechanism. In section 2, I present my baseline model. In section 3, I derive equilibrium and explore its properties. In section 4, I consider the welfare properties of the equilibrium and characterize optimal policy. In section 5, I conclude.

Literature. This paper is part of the literature that explores amplification mechanisms that operate through credit channels, and their role in macroeconomic fluctuations. The most common approach in this literature is to consider constraints on firms' ability to borrow to finance new investment. Important early papers in this literature include [Bernanke, Gertler, and Gilchrist \(1999\)](#), [Kiyotaki and Moore \(1997\)](#), and [Carlstrom and Fuerst \(1997\)](#). In these papers, a fall in the value of firm collateral can cause binding constraints, reducing investment and future production, and thus further reducing the value of collateral.

This paper represents a substantial departure from this tradition by shifting the focus to consumer credit. This is a significant difference because the precise nature of the amplification process may yield different predictions about the dynamics of macroeconomic variables, and quite different policy recommendations. For instance, if household credit is the primary mechanism, then the distribution of household debt is an important macroeconomic parameter and should be monitored and possibly regulated by economic authorities.

A number of recent empirical papers have supported the idea that consumer credit may be a key component in the 2008 recession in the U.S. [Hall \(2011a\)](#) argues that the wedge between the rate at which households save and firms borrow is important in explaining the recent recession. [Hall \(2011b\)](#) argues that the fall in consumer demand due to reduced credit access played an important role in the 2008 recession. In a series of papers, (i.e. [Mian, Rao, and Sufi \(2011\)](#), [Mian and Sufi \(2010\)](#), [Mian and Sufi \(2011\)](#), [Mian and Sufi \(2012\)](#)), Mian and Sufi document that the severity of the 2008 recession varied systematically across counties by consumer leverage, housing price declines, and drops in consumer demand. Their work suggests a substantial consumer demand channel, and that this channel depends critically on credit access, indebtedness, and housing prices.

Motivated in part by these findings, as well as the rise in consumer leverage during the 2000s, a number of recent papers have examined the connection between household wealth, consumer credit, and the business cycle. [Campbell and Hercowitz \(2005\)](#) and [Campbell and Hercowitz \(2009\)](#) explore the consequences for the U.S. economy of the relaxation of equity requirements for household borrowing. They find that these innovations reduced aggregate volatility, but may harm borrowers in the long run by raising interest rates. [Iacoviello \(2005\)](#) explores the effects of collateralized borrowing and housing prices on the business cycle, and finds some mild amplification. [Midrigan and Philippon \(2011\)](#) impose a cash-in-advance constraint on households that can be relaxed by collateralized borrowing, which is quite similar to my own setup, but the details of the model are quite different and the focus is not on the amplification effects that I describe.

My paper is also closely related to what I refer to as models of aggregate demand. Models in this literature share the common feature that a reduction in demand from some agents in the economy reduce the profitability of others. Thus the economy may exhibit fluctuations that are primarily or entirely the result of coordination failures between agents in the economy. Many such models depend heavily on decentralized exchange. A classic example is [Diamond \(1982\)](#), in which the decision of agents to participate in markets for exchange depends on the probability of making an exchange, which is increasing in the fraction of agents that produce. Thus there are multiple equilibria and amplification effects. More recent papers in this literature include [Chamley \(2012\)](#) and the above-cited [Guerrieri and Lorenzoni \(2009\)](#). Another approach is to rely on sticky prices to generate aggregate demand effects. Recent examples in this style include [Eggertsson and Krugman \(2012\)](#) and [Cúrdia and Woodford \(2010\)](#). Finally, many models rely on borrowing constraints in a heterogeneous agent framework, following the classic papers [Bewley \(1977\)](#) and [Aiyagari \(1994\)](#). A recent example in this literature is [Guerrieri and Lorenzoni \(2011\)](#).

The papers closest to my approach are [Guerrieri and Lorenzoni \(2009\)](#) and [Guerrieri and Lorenzoni \(2011\)](#). [Guerrieri and Lorenzoni \(2009\)](#) explore the role of precautionary motives of households in amplifying fluctuations in a three-period model. Although the model focuses on money rather than borrowing, the structure of the model is nearly isomorphic to mine. However, by focusing on borrowing rather than money, I introduce the possibility of an amplification process driven by factors beyond the precautionary motive. [Guerrieri and Lorenzoni \(2011\)](#) aim to extend this basic model to an infinite-horizon setup and focus on the transition paths resulting from an exogenous change in borrowing constraints. I depart from this model by introducing the cash-in-advance setup, which is the most important component of [Guerrieri and Lorenzoni \(2009\)](#) that did not make the

transition, and endogenize the borrowing constraint by making it dependent on collateral value.

2 Three Period Model

I first analyze the main mechanism in a tractable three-period model, with periods denoted by $t = 0, 1, 2$. There are three types of agents: households, entrepreneurs, and banks. Households produce and buy and sell from one another, and are subject to idiosyncratic labor productivity shocks. Banks accept deposits from households, and make loans to households and entrepreneurs. Entrepreneurs borrow to finance their projects.

Households. There is a unit measure of households with lifetime utility

$$U = E_0 \left[u(c_{i0}) + \beta [u(c_{i1}) - v(n_{i1})] + \beta^2 [u(c_{i2}) + \phi W_{i2}] \right] \quad (2.1)$$

where c_{it} and n_{it} are consumption and labor supply of household i in period t respectively, and W_{i2} is household i 's net wealth at the end of period 2.⁴ I assume that $u(\cdot)$ is strictly increasing, strictly concave, and satisfies $u'(c) \rightarrow \infty$ as $c \rightarrow 0$ and $u'(c) \rightarrow 0$ as $c \rightarrow \infty$. I assume that $v(\cdot)$ is strictly increasing and strictly convex, and satisfies $v'(n) \rightarrow 0$ as $n \rightarrow 0$, and $v'(n) \rightarrow \bar{n}$ as $n \rightarrow \infty$, where \bar{n} is the maximum possible supply of labor. I further assume that $v(\cdot)$ satisfies $v''(n) < [v'(n)]^2/v(n)$ for all n .⁵

In period 0, households receive endowments y_{i0} , which are drawn from finite set $\{y_{\min}, \dots, y_{\max}\}$. Taking the interest rate R_1 as given, households choose consumption c_{i0} , and enter period 1 with net asset position

$$a_{i1} = R_1 (y_{i0} - c_{i0}) \quad (2.2)$$

In period 1, households learn their idiosyncratic productivities θ_{i1} , which are drawn from finite set $\{\theta_{\min}, \dots, \theta_{\max}\}$, and the aggregate productivity shock z_1 . They exert effort n_{i1} to produce consumption goods $y_{i1} = z_1 \theta_{i1} n_{i1}$. Then households choose consumption c_{i1} , and taking the interest rate R_2 as given, they enter the final period with assets

$$a_{i2} = R_2 (a_{i1} + y_{i1} - c_{i1}) \quad (2.3)$$

⁴By making preferences quasilinear in end-of-period wealth, I use a technique common in models of monetary exchange such as [Lagos and Wright \(2005\)](#) and [Guerrieri and Lorenzoni \(2009\)](#).

⁵This assumption says that $v(n)$ everywhere grows more slowly than an exponential function.

Since households cannot consume their own produce, they must purchase any consumption in excess of a_{i1} from other households. However, since they have not yet received the income from the sale of their own output at the time of purchase, they must have sufficient funds on hand. Households may borrow from banks on an intraperiod loan market to finance these purchases, subject to a borrowing constraint set by banks. Banks cannot observe a household's productivity, labor effort, or assets, and set borrowing constraint \bar{b}_1 that applies to all households. Therefore a household's period 1 consumption is limited by

$$c_{i1} \leq a_{i1} + \bar{b}_1 \quad (2.4)$$

Entering period 2, households receive endowment y_{i2} , choose consumption c_{i2} , and end the period with wealth

$$W_{i2} = a_{i2} + y_{i2} - c_{i2} \quad (2.5)$$

Households receive a fraction γ of their period 2 income before they consume, and the rest afterward; they can borrow against their future income on the intraperiod credit market, facing exogenous constraint \bar{b}_2 . Thus households face a period 2 borrowing constraint

$$c_{i2} \leq a_{i1} + \bar{b}_2 + \gamma y_{i2} \quad (2.6)$$

Altogether, household i maximizes (2.1) subject to constraints (2.2) - (2.6). Suppressing the subscript i , the first-order conditions of this problem are

$$u'(c_0) = \beta R_1 E_0 [u'(c_1)] \quad (2.7)$$

$$u'(c_1) = \beta R_2 E_1 [u'(c_2)] + \mu \quad (2.8)$$

$$v'(n_1) = z_1 \theta_1 \beta R_2 E_1 [u'(c_2)] \quad (2.9)$$

$$u'(c_2) = \phi + \lambda \quad (2.10)$$

where μ and λ are the lagrange multipliers on the period 1 and 2 intraperiod borrowing constraints respectively. Additionally, the period budget constraints (2.2), (2.3), and (2.5) hold with equality, and we have complementary slackness conditions

$$\mu \cdot (a_1 + \bar{b}_1 - c_1) \geq 0 \quad (2.11)$$

$$\lambda \cdot (a_2 + \gamma y_2 + \bar{b}_2 - c_2) \geq 0 \quad (2.12)$$

Equation (2.8) is the household Euler equation for period 1. When the constraint binds, $\mu > 0$ and the household consumes less than its preferred level. However, since current income cannot be used to directly finance current consumption, binding constraints do

not affect the choice of labor supply. We can see this in that μ does not appear in equation (2.9).

Combining (2.8) and (2.9), we obtain the labor-leisure tradeoff

$$v'(n_1) = z_1 \theta_1 \left(1 - \frac{\mu}{u'(c_1)} \right) u'(c_1) \quad (2.13)$$

When constraints do not bind, $\mu = 0$ and we obtain a standard labor-leisure expression. However, when constraints bind, $\mu > 0$ and binding constraints act as a labor wedge, resulting in undersupply of labor relative to the optimal level.

Entrepreneurs. There is a unit measure of entrepreneurs born in period 0 and in period 1. Entrepreneurs born in period t have linear utility $U_t = c_t + c_{t+1}$. Entrepreneur i born in period t receives a project opportunity, which is a tree that will yield return $R_{i,t+1}$ in period $t + 1$ for up to \bar{I} investment in period t .

The returns of period t entrepreneur projects are distributed with cdf $\iota_t(\cdot)$. Entrepreneurs can borrow funds at gross interest rate R_{t+1} , and so entrepreneur i will choose to fully finance his project as long as $R_{i,t+1} \geq R_{t+1}$. This defines an investment demand curve

$$I_t(R_{t+1}) = (1 - \iota_t(R_{t+1}))\bar{I} \quad (2.14)$$

Entrepreneurs of generation t will repay $R_{t+1}I_t(R_{t+1})$ to their creditors in period $t + 1$, and will consume the remainder of their output. Therefore entrepreneur aggregate consumption, which is equivalent to welfare given their linear utility, is

$$C_{t+1}^{ent} = \int_{R_{t+1}}^{\infty} I_t(R) dR$$

Banks. There is a unit measure of banks, which make loans to and accept deposits from households and entrepreneurs. Banks are perfectly competitive and make zero profits in equilibrium. They also have perfect credibility, so that their deposit slips circulate as currency.

2.1 Borrowing Constraint

I assume that banks set the borrowing constraint such that no household defaults in equilibrium. When a household defaults, the bank seizes the household's assets a_{i2} , and also imposes a utility penalty equal to a loss of χR_2 in end of period 2 assets. This latter cost

reflects the long-term consequences of default beyond immediate loss of assets, for instance from reputational effects. This penalty is increasing in the return to production, which reflects that the reputational costs of default are greater in good times than in bad.

Households choose whether to repay their debt b_{i1} at the end of period 1. However, they learn all information relevant to this decision at the beginning of period 1. Thus a household that intends to default will choose period 1 labor and consumption with the expectation that end of period wealth will be seized. Suppressing the subscript i , we can write the household value function under default as:

$$\begin{aligned} V^d(a_1, \bar{b}_1) &= \max_{c_1^d, n_1^d, c_2^d} \left\{ u(c_1^d) - v(n_1^d) + \beta[u(c_2^d) + \phi(y_2 - c_2^d) - R_2\phi\chi] \right\} \\ \text{s.t. } c_1^d &\leq a_1 + \bar{b}_1 \\ c_2^d &\leq \gamma y_2 + \bar{b}_2 \\ n_1^d &\geq 0 \end{aligned}$$

From first-order conditions we have immediately that $c_1^d = a_1 + \bar{b}_1$ and $n_1^d = 0$. Intuitively, since defaulting households anticipate the seizure of their assets, they will not bother to work and will consume all available funds. Thus the value function under default can be written

$$V^d(a_1, \bar{b}_1) = u(a_1 + \bar{b}_1) + \beta E_1 V_2(0) - \beta \phi R_2 \chi$$

where $V_2(a)$ is the value function of a household entering period 2 with assets a . We can likewise write the value function under repayment as

$$V^R(\cdot) = u(c(\cdot)) - v(n(\cdot)) + \beta E_1 V_2(a_2(\cdot))$$

where $c(\cdot)$, $n(\cdot)$, and $a_2(\cdot)$ are the household decision rules defined by (2.7) – (2.12).

A household will repay as long as $V^R \geq V^d$, and so banks will set \bar{b}_1 so that

$$V^R(\bar{b}_1, \cdot) \geq V^d(\bar{b}_1, \cdot), \forall i \tag{2.15}$$

Further, competition among banks for borrowers will push the constraint up until some household is just indifferent between defaulting and repaying, or all households are unconstrained. Thus equilibrium \bar{b}_1 will satisfy

$$V^R(\bar{b}_1, \cdot) = V^d(\bar{b}_1, \cdot), \text{ for some } i \quad \text{or} \quad \mu_i = 0, \forall i \tag{2.16}$$

From (2.16), we see that there are two cases to consider. First, it may be that some household is constrained in equilibrium, in which case $V^R = V^d$ for this household defines \bar{b}_1 implicitly, and (as we shall see below) uniquely. Second, it may be that all households are unconstrained in equilibrium and none choose to default. In this case, there may be many values of \bar{b}_1 that are consistent with both (2.15) and (2.16). We can choose any of these values without affecting equilibrium, and so we shall choose one that yields a tractable expression for \bar{b}_1 .

To do so by defining an alternate value function V^*

$$\begin{aligned} V^*(a_1, \theta_1, R_2, z, \bar{b}_1) &= \max_{n_1^*, c_2^*} \{u(a_1 + \bar{b}_1) - v(n^*) + \beta[u(c_2^*) + \phi(y_2 + a_2^* - c_2^*)]\} \\ \text{s.t. } a_2^* &= R_2(z_1\theta_1 n_1^* - \bar{b}_1) \\ c_2^* &\leq \gamma y_2 + a_2^* + \bar{b}_2 \end{aligned}$$

V^* is the value function under repayment of a household that is forced to consume all available funds in period 1, i.e. $c_1 = a_1 + \bar{b}_1$. Trivially, when a household is constrained under repayment, we have $V^* = V^R$. However, when a household is unconstrained under repayment, we will in general have $V^R > V^*$.

V^* turns out to be a useful construct because the utility from period 1 consumption is the same as under default, and so the expression $V^* - V^d$ will not depend on period 1 assets a_1 . Moreover, since unconstrained households are indifferent to the precise level of \bar{b}_1 , we can set \bar{b}_1 as though all households were constrained without affecting any household's decisions.

We can define the function

$$G(\cdot) = V^*(\cdot) - V^d(\cdot)$$

where $G(\cdot)$ is the net return from repayment under binding constraints. Since $V^* \geq V^R$, a household will repay if $G(\cdot) > 0$ for that household. Now we establish a few useful properties of $G(\cdot)$.

Lemma 1. *The function $G(a_1, \theta_1, R_2, z_1, \bar{b}_1)$,*

1. *does not depend on a_1 .*
2. *is strictly decreasing in \bar{b}_1 .*
3. *is strictly increasing in z_1 and θ_1 .*

Proof. See Appendix 1. □

Since $G(\cdot)$ is strictly decreasing in \bar{b}_1 , for each household i there will be unique breakeven level of borrowing \bar{b}_i at which $G(\bar{b}_i, \cdot) = 0$. Then we can simply assume that banks set $\bar{b}_1 = \min\{\bar{b}_i\}$, which will satisfy both (2.15) and (2.16). Since households differ only in a_1 and θ_1 , and since $G(\cdot)$ does not depend on a_1 and is strictly increasing in θ_1 , this is equivalent to choosing \bar{b}_1 such that

$$G(\bar{b}_1, \theta_{\min}) = 0 \quad (2.17)$$

To obtain an expression for \bar{b}_1 , consider a household of the lowest productivity type $\theta_{i1} = \theta_{\min}$. Then we define n_1^* as the labor supply in the (*) case, and c_2^* and c_2^d as period 2 consumption under default and the (*) case respectively. Then \bar{b}_1 satisfies

$$\bar{b}_1 = z_1 \theta_{\min} n_1^* - \frac{v(n_1^*)}{\beta \phi R_2} + \frac{\Delta^* - \Delta^d}{\phi R_2} + \chi \quad (2.18)$$

where $\Delta^* = u(c_2^*) - \phi c_2^*$ and $\Delta^d = u(c_2^d) - \phi c_2^d$.

The right hand side of (2.18) contains three terms, which correspond to the three benefits of repayment. The first benefit is that the repaying household is able to save, and so enjoys the surplus of its wage earnings above the disutility from labor $\beta \phi R_2 z_1 \theta_{\min} n_1^* - v(n_1^*)$. The second benefit is that the repaying household enters period 2 with greater assets, which may be used for intra-period 2 consumption smoothing. If $\bar{b}_2 + \gamma y_2$ is less than that level of c that attains $u'(c) = \phi$, there is additional benefit to entering with positive assets because it allows this constraint to be relaxed. Finally, repayment avoids the reputational penalty χ .

We next calculate how \bar{b}_1 depends on other variables by applying the implicit function theorem to (2.17).

$$\begin{aligned} \frac{d\bar{b}_1}{dR_2} &= \frac{z_1 \theta_{\min} n_1^* - \bar{b}_1 + \frac{\phi}{u'(c_2^*)} \chi}{R_2} \\ \frac{d\bar{b}_1}{dz_1} &= \theta_{\min} n_1^* \\ \frac{d\bar{b}_1}{d\theta_{\min}} &= z_1 n_1^* \end{aligned} \quad (2.19)$$

Using the expression (2.18), we can write the numerator of (2.19) as

$$\frac{v(n_1^*)}{\beta\phi R_2} - \left(\frac{\Delta^* - \Delta^d}{\phi R_2} \right) - \left(1 - \frac{\phi}{u'(c_2^*)} \right) \chi$$

which will surely be positive if the period 2 constraint does not bind. If the period 2 constraint does bind, this expression might be negative.

We also find that the borrowing constraint is increasing in the individual productivity of the lowest type household θ_{\min} , and the aggregate productivity z_1 . This is also intuitive, because a more productive household would like to produce more, and therefore save more, making exclusion from saving more costly.

3 Equilibrium

To define equilibrium, we must add a resource constraint to each period.⁶ The simplest way to express this is as market clearing in the asset market, in which total real savings of households S_t equals investment of entrepreneurs I_t

$$S_t = \int_i \frac{a_{i,t+1}}{R_{t+1}} = I_t(R_{t+1}) \quad (3.1)$$

In an application of Walras' Law, if we substitute the household period budget constraints (2.2) and (2.3) into (3.1), we would obtain an expression for market clearing in the period t goods market

$$Y_t = C_t + I_t - A_t$$

where $A_t = \int_i a_{it}$ are total initial net assets in the economy.

We are now in position to define equilibrium:

Definition 1 (Equilibrium in Three Period Model). Given an initial distribution of income $\{y_{i0}\}$, realizations of z_1 and $\{\theta_{i1}\}$, investment schedules $i_0(\cdot)$ and $i_1(\cdot)$, \bar{b}_2 and $\{y_{i2}\}$, equilibrium $\{R_1, R_2, \bar{b}_1, c_{i0}, c_{i1}, c_{i2}, n_{i1}, a_{i1}, a_{i2}, \mu_i, \lambda_i, W_{i2}\}$ satisfy (2.2), (2.3), (2.5), and (2.7) - (3.1).

I now characterize the equilibrium using backward induction.

⁶Since household utility is quasilinear in W_{i2} , there is no resource constraint for period 2.

3.1 Period 1 Equilibrium ($\lambda = 0$)

Consider the period 1 equilibrium for some initial state $\{a_{i1}, \theta_{i1}, z_t\}$. We begin by analyzing the simpler case in which the period 2 constraint never binds, i.e. $\lambda = 0$, and $u'(c_{i2}) = \phi$ for all households.⁷

Household i 's labor supply is defined by

$$v'(n_{i1}) = z_1 \theta_{i1} \beta R_2 \phi$$

and aggregate production of consumption goods can be expressed as

$$Y_1(R_2) = z_1 \int_i \theta_{i1} n_{i1}$$

This defines an aggregate supply curve that is upward-sloping in R_2 .

From (2.8), household i 's *unconstrained* consumption is defined by

$$u'(c_{i1}^u) = \beta R_2 \phi$$

which is decreasing in R_2 . However, if desired consumption exceeds available funds $a + \bar{b}$, household i will be forced to consume less than its desired amount. Thus household i 's actual consumption is

$$c_{i1}(R_2) = \min(c_{i1}^u, a_{i1} + \bar{b})$$

The only other source of demand for currently produced goods is from entrepreneurs, who desire real goods to invest in their projects. Thus the aggregate demand for output can be expressed as a function of R_2

$$D_1(R_2) = I_1(R_2) + \int_i c_{i1}(R_2) - A_1$$

where $A_1 = \int_i a_{i1}$. Note that while investment demand I_1 and unconstrained desired consumption are both decreasing in R_2 , the borrowing constraint \bar{b} is *increasing* in R_2 , and so aggregate demand may be increasing over some interval.

We can depict the period 1 equilibrium using an aggregate supply and demand diagram. Figure 1 depicts the equilibrium that would prevail if there were no borrowing constraint. The aggregate demand curve is decreasing everywhere, and there is necessarily a unique equilibrium.

⁷This assumption is equivalent to $u'(\bar{b}_2 + \gamma y_2 + a_{i2}) \leq \phi$ for all households i .

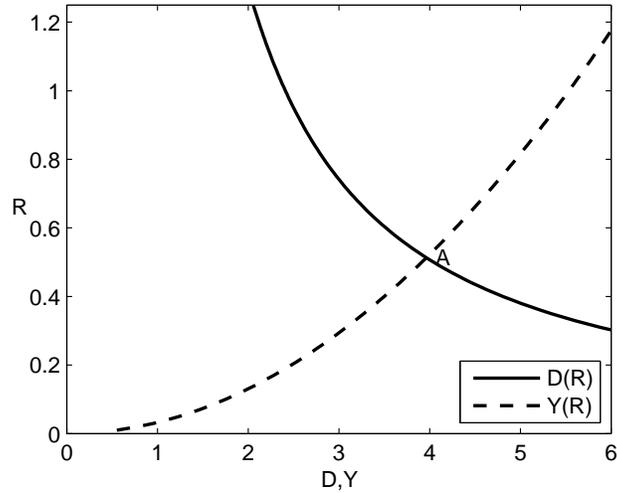


Figure 1: Unconstrained Equilibrium in the Three Period Model

Now consider the effect of a decrease in aggregate productivity, equivalent to a negative supply shock in a standard growth model. This produces a leftward shift in the aggregate supply curve, which is depicted in figure 2. At the original equilibrium interest rate, aggregate demand is greater than supply. To restore equilibrium the interest rate rises, increasing labor supply and decreasing consumption of goods.

Note that because households cannot consume their own output, this process is mediated through the interest rate. The higher interest rate signals scarcity of goods today relative to tomorrow, prompting households to increase labor effort and reduce consumption in order to take advantage of the higher interest rates.

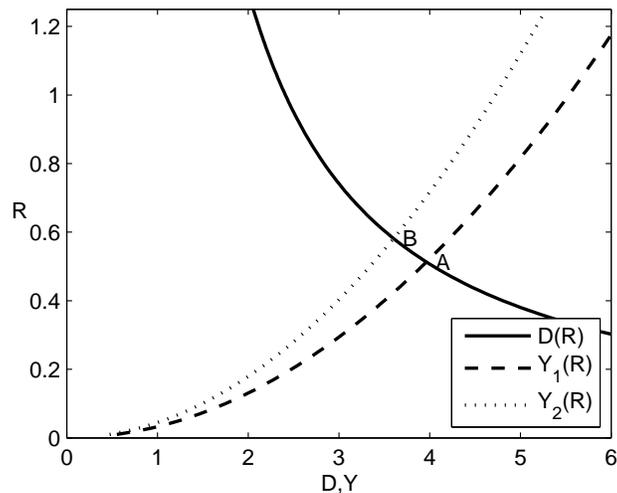


Figure 2: Productivity Shock in the Unconstrained Three Period Model

This discussion also illustrates an important prediction of the standard growth model: real business cycles result in countercyclical real interest rates. Nor does this prediction hinge on the absence of capital: If there were capital, the economy would reduce investment and (assuming no irreversibility) consume capital goods. This would reduce capital in the future, increasing the marginal product of capital and the interest rate.⁸

Next we consider an economy in which the borrowing constraint binds for some households for some values of R_2 . Figure 3 depicts such a case. In this example, there is an equal share of households with five initial wealth levels $a_{i1} \in \{a_1, \dots, a_5\}$. Half of households have low productivity θ_L , and half have high productivity $\theta_H > \theta_L$. Productivity is independent of wealth.

For sufficiently high R_2 , the constraint is loose for all households, and so the aggregate demand curve is the same as in Figure 1. However, as R_2 decreases, desired consumption increases and eventually the constraint binds, first for the lowest wealth households, and then for progressively higher wealth households. This results in the aggregate demand curve becoming steeper, and eventually upward sloping when all households are constrained.

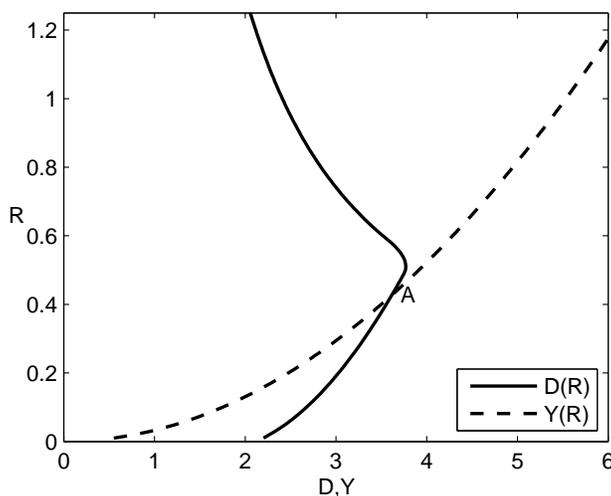


Figure 3: Three Period Model with Constrained Aggregate Demand

In Figure 3, constrained consumption is strictly increasing in \bar{b} . In fact, this is always the case:

Lemma 2. *If period 2 consumption is not constrained, then \bar{b} is increasing in R_2 .*

⁸This is for a supply shock without persistence. A persistent supply shock might lower the marginal product of capital in the next period enough so that interest rates would be procyclical.

Proof. See Appendix 1. □

Moreover, we can establish that as R_2 decreases, all households become constrained:

Lemma 3. *As $R_2 \rightarrow 0$, desired unconstrained consumption goes to ∞ , the borrowing constraint \bar{b} goes to 0, and the constraint binds for all households.*

Proof. See Appendix 1. □

Lemma 2 and 3 immediately lead to:

Corollary 1. *The slope of $C(R)$ is always greater when the constraint binds for some households than when the constraint does not bind, and is strictly positive over some region.*

Proof. See Appendix 1. □

We are now in position to prove the following useful proposition:

Proposition 1 (Interest Rate Cutoff). *For each household i , there exist a level of interest rate $\bar{R}_i > 0$ such that for all $R \geq \bar{R}_i$ the household is unconstrained, and for $R < \bar{R}_i$ the household is constrained. \bar{R}_i is strictly increasing in a_{i1} , but does not depend on θ_{i1} .*

Proof. See Appendix 1. □

Proposition 1, together with Lemmas 2 and 3, imply that the constrained consumption curve is upward sloping for sufficiently small R . As R increases, the constraint loosens for household, decreasing in slope, becoming downward sloping, and eventually becoming identical to the unconstrained consumption curve for sufficiently high R . That is, the consumption demand curve will have a shape similar to the aggregate demand curve in figure 3.

Since the aggregate demand curve also includes the investment demand $I(R)$, which is everywhere decreasing in R , we haven't established that the aggregate demand curve will have this shape. However, if we assume that investment demand is fairly inelastic, the shape will be determined primarily by consumption demand.

Contrasting the equilibria depicted in Figures 1 and 3, we see that output is strictly lower and the interest rate strictly higher with the borrowing constraint than without it. In fact, this will be true of anything that decreases the aggregate demand curve.

Proposition 2 (Lower Aggregate Demand). *If two economies have the same aggregate supply function $Y(R)$, both have a unique equilibrium, and one has lower aggregate demand $D_1(R) \leq D_2(R)$, then the economy with lower aggregate demand will have lower equilibrium output and real interest rate, and strictly lower if $D_1(R) < D_2(R)$ at the initial equilibrium R .*

Proof. See Appendix 1. □

Corollary 2 (Effect of Borrowing Constraint). *Equilibrium output and the real interest rate are always lower in an economy with a period 1 borrowing constraint than in an otherwise identical economy with no borrowing constraint, and strictly lower if any constraint is binding in equilibrium.*

Proof. See Appendix 1. □

Response to Shocks. We next consider the effect of shocks to various parameters, and how these differ when borrowing constraints are binding. We can decompose such shifts into three types: those affecting the aggregate supply curve only, those affecting only the aggregate demand curve, and those affecting both. Of the first type are shifts in household productivity θ , excepting the lowest such productivity, and shifts in the elasticity of leisure ζ . Of the second type are changes in the distribution of assets a_1 , and shifts in the investment demand schedule. Of the last type are shifts in aggregate productivity z , or the idiosyncratic productivity of the lowest productivity households θ_{\min} .

Introducing borrowing constraints tends to *dampen* shifts of the aggregate supply curve, and to *amplify* shifts of the aggregate demand curve. This is because the aggregate demand curve is steeper when constraints are binding, as we saw in Corollary 1.

To see why a steeper aggregate demand curve implies dampening of supply shocks, consider a marginal shift in the aggregate supply curve dAS . Such a shift is depicted in Figure 4. We can see geometrically that the resulting shift in equilibrium output is a simple function of the slopes of the AD and AS curves:

$$\frac{dY}{dAS} = \frac{m}{m+n}$$

where $m = 1/Y'(R)$ is the slope of the AS curve, and $n = -1/D'(R)$ is minus the slope of the AD curve.

The term $\frac{m}{m+n}$ is the marginal effect on equilibrium output arising from a shift in aggregate supply. When the constraint is not binding, $Y'(R) > 0$ and $D'(R) < 0$, so that $\frac{m}{m+n} \in (0, 1)$. When the constraint binds for some households, the AD curve becomes steeper, i.e. n becomes larger, and $\frac{m}{m+n}$ becomes *smaller*, so that supply shocks are dampened.

Not only does the presence of a constraint dampen supply shocks, it may even reverse their sign. This is because the AD curve is upward sloping for some R . In such a region, we have $n < 0$, and $m+n < 0$ because $D'(R) < Y'(R)$. Thus the *sign* of this relation

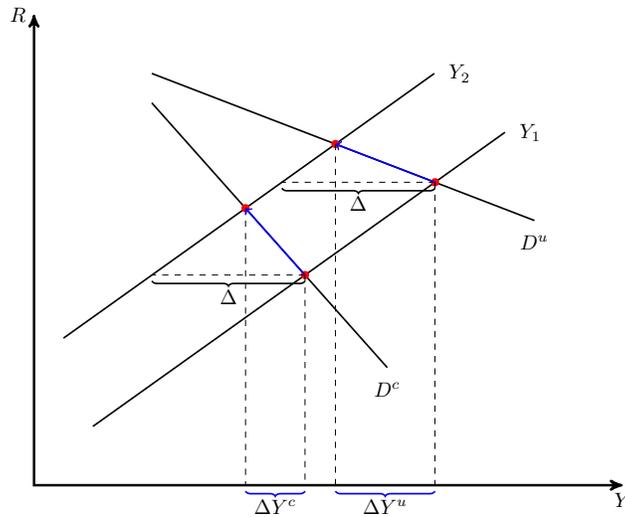


Figure 4: Stylized Aggregate Supply Shift

reverses, and a negative supply shock brings about an *increase* in output. This case is depicted in figure 5.

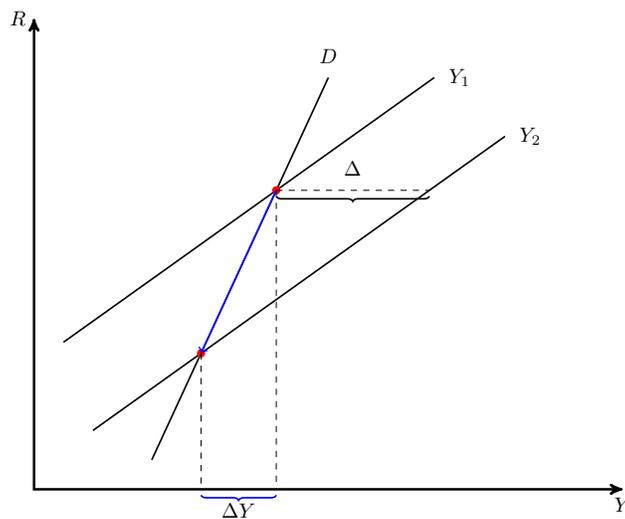


Figure 5: Paradox of Toil

Intuitively, an increase in willingness to work when most consumers are borrowing constrained is self-defeating, because there is insufficient borrowing capacity to finance consumption of the greater supply of output, and so the increase in labor supply will drive down the equilibrium interest rate, causing borrowing constraints to tighten, and thus decreasing equilibrium output. This is analogous to the *paradox of toil* observed in New Keynesian models with a zero lower bound, such as [Eggertsson \(2010\)](#) and [Eggertsson and Krugman \(2012\)](#).

Now we turn to the case of a negative *demand* shock. Consider a leftward shift of the AD curve of dAD , as depicted in figure 6. This results in a fall in output equal to

$$\frac{dY}{dAD} = \frac{n}{m+n}$$

The term $\frac{n}{m+n}$ is the marginal *multiplier* applied to any exogenous shift in aggregate demand. When the constraint is not binding the multiplier is less than 1. When the constraint binds for some households, the AD curve becomes steeper, meaning that n becomes larger. This causes the multiplier to increase. When sufficiently many households become constrained, the AD curve becomes upward sloping. Then n is negative and greater than m , and so $\frac{n}{m+n} > 1$, i.e. the multiplier is greater than 1.

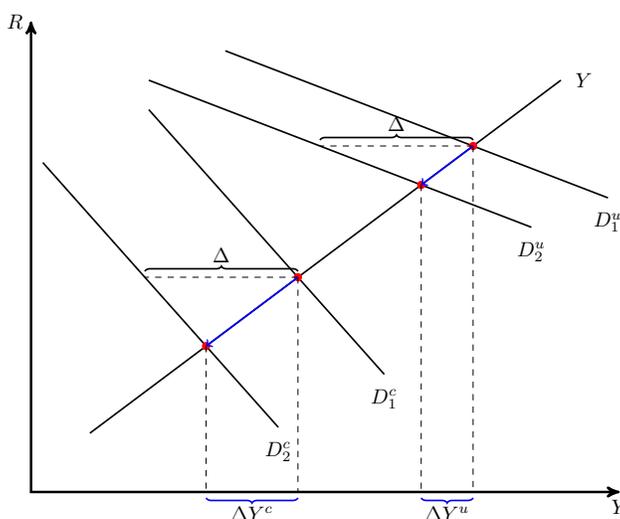


Figure 6: Stylized Aggregate Demand Shift

Following the logic above, we can derive a simple analytical expression for the marginal effect of a shock dx .

Proposition 3 (Amplification and Dampening of Shocks). *The effect on output Y of a marginal change in parameter x is given by*

$$\frac{dY}{dx} = \left(\frac{-D'(R)}{Y'(R) - D'(R)} \right) \frac{\partial Y}{\partial x} + \left(\frac{Y'(R)}{Y'(R) - D'(R)} \right) \frac{\partial D}{\partial x} \quad (3.2)$$

Holding all else fixed, decreasing $D'(R)$ will increase $\frac{dY}{dx}$ iff $\frac{\partial D}{\partial x} \geq \frac{\partial Y}{\partial x}$.

Proof. See Appendix 1. □

Proposition 3 suggests a simple standard for determining whether a shock will be amplified or dampened by the introduction of the constraint: whether the shift in the aggregate supply or aggregate demand schedule is greater. Clearly any shock that shifts only one of these two will have an unambiguous outcome, but this allows a simple test to be applied in ambiguous cases.

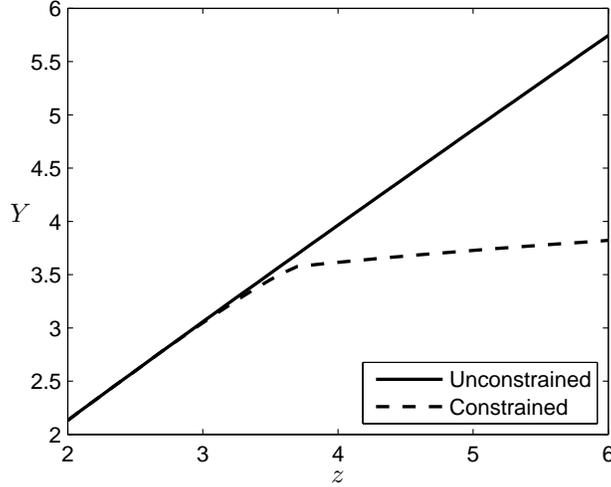


Figure 7: Output for varying levels of aggregate productivity

Now we apply this test to the case of a change in z . Note that a fall in z will not only shift the AS curve inward, but will also shift in the aggregate demand curve in the constrained region. This is because the borrowing constraint \bar{b} is increasing in z . However, we can show that the inward shift in the AS curve due to a fall in z is always *larger* than the outward shift in the AD curve.

To see this, observe that the marginal change in the AS curve is

$$\frac{\partial Y}{\partial z} = \int_i \theta_i n_i + \int_i \frac{\theta_i^2 z \beta \phi R}{v''(n_i)}$$

whereas the change in the AD curve is

$$\frac{\partial D}{\partial z} = \int_{i \in C} \theta_{min} n_{min}$$

Since the term $\int_i \theta_i^2 z \beta \phi R [v''(n_i)]^{-1} > 0$, and the term $\int_i \theta_i n_i$ is necessarily greater than $\int_{i \in con} \theta_{min} n_{min}$ because $\theta_i \geq \theta_{min}$, $n_i \geq n_{min}$, and the latter integral is over a subset of households.

Figure 7 depicts this dampening in the case of fluctuations in z . The two lines depict output for various levels of aggregate productivity z for two economies: one with the constraint and one without. For low values of z , the constraint is loose for most households and so output is about the same in both economies. For high levels of z , the constraint binds and so increasing z will increase output little if at all. In the case depicted, the curve levels off because the negative effect on output from the paradox of toil is just offset by the outward shift in the borrowing constraint produced by higher productivity.

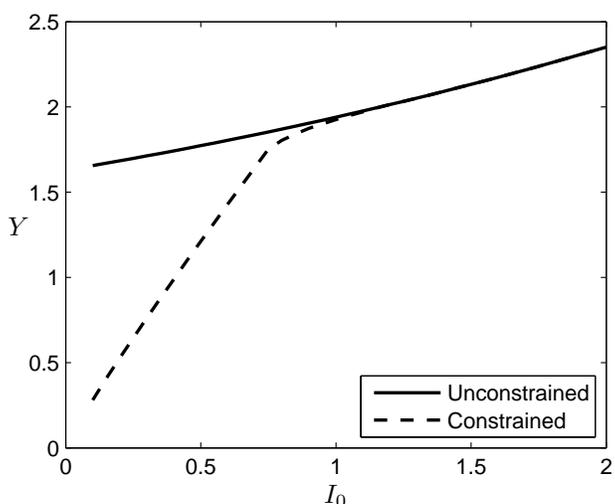


Figure 8: Output for varying levels of Investment Demand

By contrast, the presence of the constraint will tend to amplify shifts in investment demand, initial assets of households, and in some cases shifts in θ_{\min} . An example of amplification is shown in figure 8, which depicts the effect of shifts in the investment demand schedule. In particular, I consider shifts in a linear investment demand schedule with x -intercept I_0 . For high levels of investment demand, the constraint is not binding for most households, and so the behavior of the economies with and without the constraint are nearly identical. However, if investment demand falls too low, the lower interest rates will cause borrowing constraints to bind for households, causing further declines in demand and amplification.

3.2 Period 1 Equilibrium ($\lambda \geq 0$)

Now we relax the assumption that the period 2 constraint never binds. Let $\bar{m} = \gamma y_2 + \bar{b}_2$ be new funds available at the beginning of period 2. Then any household that enters period 2 with assets a_{i2} that satisfy $u'(\bar{m} + a_{i2}) > \phi$ will experience binding constraints in period 2.

Any household for whom the period 2 constraint binds will increase its labor supply in period 1, and will decrease its unconstrained consumption demand. By itself, these effects will generally result in a *decrease* in output, as found in e.g. [Guerrieri and Lorenzoni \(2011\)](#). However, binding period 2 constraints will tend to increase \bar{b} , by increasing the value of asset markets access between periods 1 and 2. Thus period 2 constraints will tend to *increase* constrained consumption demand.

These effects are analyzed in Lemma 4 and Proposition 4 by considering the case of a marginal decrease in \bar{m} , leading to tightening of period 2 constraints.

Lemma 4. *A fall in \bar{m} leads to:*

- (i) *A rise in \bar{b} if $u'(\bar{m}) > \phi$.*
- (ii) *A rise in $\lambda_i = u'(c_{i2})$ for households originally constrained in period 2.*
- (iii) *An increase in the share of households constrained in period 2.*

Proof. See Appendix 1. □

Proposition 4. *A fall in \bar{m} affects consumption demand and labor supply of households as follows:*

- (i) *Households that are unconstrained in both periods will not change either their labor supply or consumption demand.*
- (ii) *Households that are constrained in period 2 will increase their labor supply.*
- (iii) *Households that are constrained in period 1 will increase their consumption demand.*
- (iv) *Households that are unconstrained in period 1 will decrease their consumption demand. If these households are constrained in period 1, their decreased consumption demand will be larger than their increase in output iff*

$$z\theta n < (\xi/\sigma) \cdot c \tag{3.3}$$

where $\xi = nv''/v'$ and $\sigma = -cu''/u'$.

Proof. See Appendix 1. □

If the period 1 constraint never binds, then the only effect will be the precautionary effect described in Proposition 4(ii). Then tightening the constraint will decrease output as long as condition (3.3) is satisfied. Since the period 2 constraint is more likely to bind for households with relatively low labor income ($z\theta n$) in period 1, these households are more

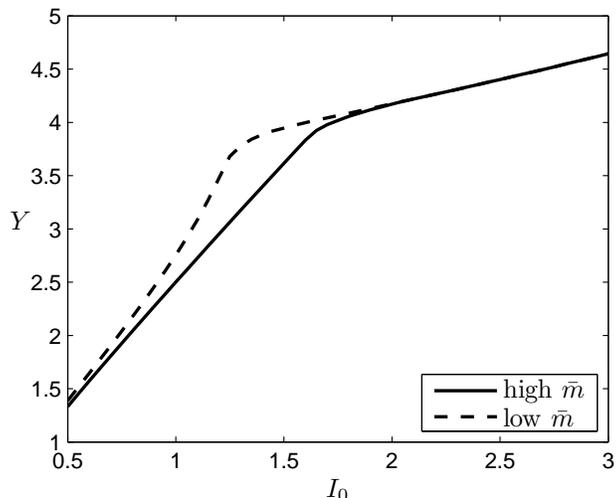


Figure 9: Output changes with period 2 constraints

likely than most to fail to satisfy (3.3). If the frisch labor elasticity ($1/\xi$) is not too large relative to the intertemporal elasticity of substitution ($1/\sigma$), and the total output of low-productivity workers is not too large relative to their consumption, then the precautionary savings effect will dominate the precautionary labor effect. For example, [Guerrieri and Lorenzoni \(2011\)](#) use the equivalent of $\xi = 1$ and $\sigma = 4$ as their baseline specification, in which case the saving effect will dominate for any household that consumes more than a quarter of their labor income in a period.

When there are households with binding period 1 constraints, the picture changes dramatically. These households will be able to *increase* their period 1 consumption when period 2 constraints tighten, because the borrowing constraint will increase. If a sufficient fraction of households are constrained in period 1, tightening period 2 constraints will increase aggregate demand, unambiguously increasing output.

The net effect of a marginal change in the period 2 available funds \bar{m} on aggregate supply Y and aggregate demand D is:

$$\frac{\partial Y}{\partial \bar{m}} = \beta R z^2 \int_{\lambda_i > 0} \theta_i^2 [v''(n_i)]^{-1} \frac{d\lambda_i}{d\bar{m}} di \quad (3.4)$$

$$\frac{\partial D}{\partial \bar{m}} = \beta R \int_{\lambda_i > 0, \mu_i = 0} [u''(c_{i1})]^{-1} \frac{d\lambda_i}{d\bar{m}} di + \int_{\mu_i > 0} \frac{d\bar{b}}{d\bar{m}} di \quad (3.5)$$

Figure 9 depicts the effect of introducing binding period 2 constraints on output for various levels of investment demand. For high levels of I_0 , few households are constrained in period 1, and so the presence of binding period 2 constraints slightly decreases

equilibrium output. For lower levels of I_0 , period 1 constraints tighten decreasing output. Now introducing period 2 constraints loosens period 1 constraints, and mitigates the fall in output. Overall, the presence of period 2 constraints decreases the responsiveness of period 1 output to fluctuations in aggregate demand.

3.3 Period 0 Equilibrium

We next turn to the determination of the period 1 initial asset distribution a_1 . In sections 3.1 and 3.2 we treated a_1 as exogenous, but it is the result of a period 0 saving decision by households.

Let $V(a_i, a, s)$ be the value function of a household entering period 1 with assets a_i , given an asset distribution a and state variables $s = (\theta_i, \theta, z)$. Then household i 's period 0 saving decision satisfies

$$u'(c_{i0}) = \beta R_1 E [V_1(a_{i1}, a_1, s)] \quad (3.6)$$

where $a_{i1} = (y_{i0} - c_{i0}) \cdot R_1$.

For a given asset distribution a_1 , (3.6) defines a unique level of consumption $c_{i0}(R_1)$ for each household. These together define total consumption $C_0(R_1) = \int_i c_{i0}(R_1)$ in the economy. Then equilibrium R_1 satisfies final goods market clearing

$$Y_0 = I_0(R_1) + C_0(R_1) \quad (3.7)$$

For a given R_1 , there is a unique period 1 equilibrium defined as in section 3.1 with period 1 assets a_1 determined by the period 0 euler equation 3.6. This defines an aggregate period 0 savings schedule

Figure 10 depicts period 0 equilibrium.

Complete Smoothing Benchmark. To characterize the period 0 equilibrium, we first suppose that the period 1 and 2 borrowing constraints never bind. Then from the period 1 and period 0 euler equations, we have $u'(c_{i1}) = \beta R_1 \phi$ and $u'(c_{i0}) = \beta R_1 E[u'(c_{i1})]$ for all households. This implies that all households have the same level of consumption in each period, i.e. $c_{i0} = \bar{c}_0$ and $c_{i1} = \bar{c}_1$.

Note that complete smoothing is a consequence of there being a fixed marginal utility of wealth at the end of period 2. Thus any variation in lifetime wealth due to differing period 0 endowments and period 2 productivity will only result in commensurate variation in end-of-model wealth W_i .

Complete smoothing is possible only when there is sufficient available funds to fi-

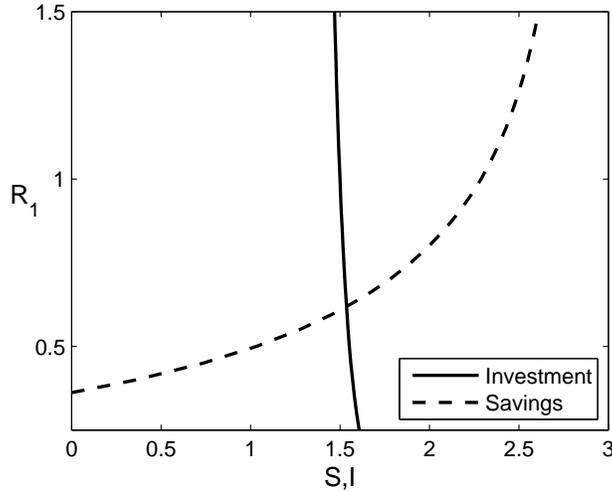


Figure 10: Period 0 Equilibrium

nance it. That is, we must have $\bar{b}_1(s) \geq \bar{c}_1(s) - a_{i1}(s)$ and $\bar{m}_2 \geq \bar{c}_2 - a_{i2}(s)$ for all states s and households i . Then the period borrowing constraints would never bind, and all households would achieve complete smoothing.

Incomplete Smoothing. Suppose that some borrowing constraint binds for some households in some states with non-zero probability. Then if there is any variation in period 0 endowments, there will be *incomplete smoothing* in period 0, meaning that high wealth households will have both higher consumption c_{i0} and higher savings a_{i1} than lower wealth households.

The reason is that consumption satisfies (3.6), and V is not linear in a_i when there is a non-zero probability of binding constraints. To see this, observe that

$$E[V_1(a_i, \cdot)] = \beta R_1 E[u'(c_{i1})] = E[\beta R_2 \phi] + R[\beta R_2 \lambda_i] + E[\mu_i]$$

Suppose that $\mu_i > 0$ in some states. Then $c_{i1} = a_{i1} + \bar{b}$, and so increasing a_{i1} will increase c_{i1} , decreasing $u'(c_{i1})$ and increasing μ_i in those states. If instead, $\mu_i = 0$ and $\lambda > 0$, then increasing a_{i1} will increase period 1 savings, increase a_{i2} , and so increase $c_{i2} = \bar{m} + a_{i2}$, decreasing λ_i .

Thus increasing a_{i1} will cause $V_1(a_{i1}, \cdot)$ to fall, leading households with higher y_{i0} to increase both consumption c_{i0} and period 1 assets a_{i1} .

Variance of Endowment. Due to incomplete smoothing, an increase in the variance of the period 0 endowment will lead to an increase in the variance of both period 0 con-

sumption c_0 and savings s_0 .

Suppose that household i 's initial endowment was distributed as a random variable y . Now consider the effect of a mean preserving spread, i.e. household i 's endowment is now $y' = y + \epsilon$, where ϵ is independent of y , with mean 0 and strictly positive variance.

Proposition 5. *A transfer of period 0 funds from households that are constrained in period 1 to households that are unconstrained in period 1 will increase desired total period 0 savings for given (R_1, R_2) . If the households receiving the transfer were originally saving more than the mean level of savings, this transfer will increase the variance of desired period 0 savings.*

Proof. See Appendix 1. □

Holding interest rates fixed, households that expect to be unconstrained in period 1 will save any increase in period 0 wealth, whereas constrained households will split this increase between period 0 and period 1. Therefore a mean-preserving spread, which transfers wealth from poor constrained households to rich unconstrained households, will lead to an increase in desired savings. This will shift out the savings curve in period 0 equilibrium, decreasing the interest rate, as depicted in Figure 11.

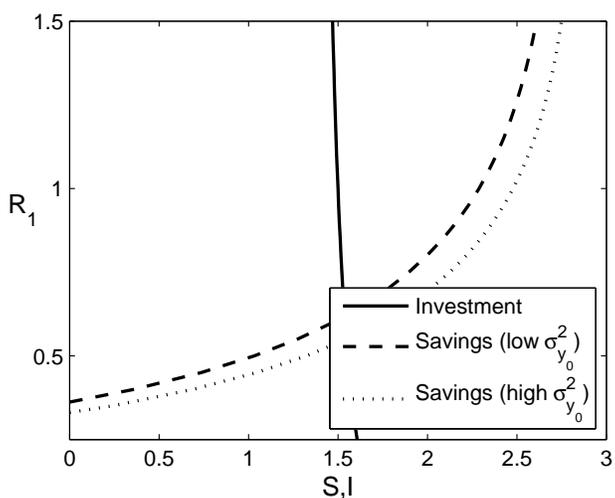


Figure 11: Period 0 Equilibrium after Mean-Preserving Spread in Endowment

An increase in the variance of the endowment will thus have two effects: to increase total savings, and to increase the variance in savings across households. This will generally result in an increase in the variance of period 1 assets, and an increase in total period 1 assets if and only if the interest elasticity of investment demand is greater than 1.

To focus on the first effect, consider a mean-preserving spread in period 1 assets a_1 . This will lead to a *decrease* in the aggregate demand curve, resulting in a fall in output and a fall in the interest rate.

Proposition 6. *A mean preserving spread in assets a_1 will shift the aggregate demand curve in, decreasing R_2 and Y_1 .*

Proof. See Appendix 1. □

To illustrate this result, suppose there were only two types of households, with initial assets a_L and $a_H > a_L$, each with measure $1/2$. Now consider a mean-preserving spread in a , so that now half of households have assets $a_L - \epsilon$ and half have $a_H + \epsilon$.

In any state and at any interest rate at which all households are constrained, or all unconstrained, this change makes no difference: In the former case, all households consume $a + \bar{b}$ regardless and so total consumption demand is $(a_L + a_H)/2 + \bar{b}$, just as in the original case, whereas in the former case, all households consume the unconstrained preferred level of consumption c^u . However, if c^u lies between $a_L + \bar{b}$ and $a_H + \bar{b}$, then the mps will lead the constrained households to decrease consumption demand by ϵ , while the unconstrained households will not change their demand. Therefore aggregate demand will fall.

Intuitively, wealthy households have a lower expected marginal propensity to consume than poorer households, and so a shift in wealth from poor to rich households will decrease aggregate demand.

Figure 12 depicts the effect of a mean-preserving spread in period 1 assets on equilibrium output for various levels of investment demand. For very high or very low demand, all households are unconstrained or constrained respectively, and the variance of the endowment makes no difference. However, when some households are constrained and some unconstrained, a mean-preserving spread decreases the interest rate and tightens the constraint, decreasing output and increasing the variance in output from negative shocks to investment demand.

4 Optimal Policy and Welfare

We now consider welfare and the effects of policy interventions. We first discuss the welfare properties of the equilibrium and derive the constrained optimum. Then we consider the effects of various policy interventions on welfare, and derive optimal policy given a restricted set of instruments. In the following discussion we restrict attention to the case without period 2 binding constraints.

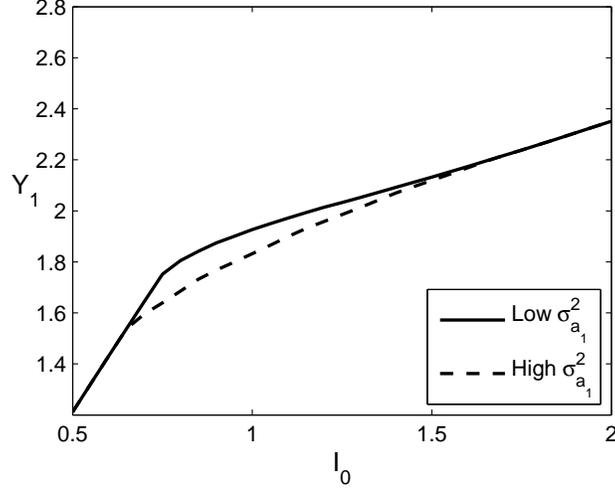


Figure 12: Period 1 Equilibrium after Mean-Preserving Spread in Assets

4.1 Welfare

The first-best corresponds to the complete smoothing benchmark we discussed in section 3.3. More interesting is the constrained optimal equilibrium.

It is again natural to break the discussion into two parts, corresponding to the period 1 and period 0 equilibria.

4.1.1 Period 1.

Consider the problem of a constrained planner that takes the initial asset distribution a_1 as given, and chooses household decisions and the interest rate subject to household borrowing constraints defined by (2.17).

The constrained planner's problem is

$$V_1^p(\{a_i\}) = \max_{\{c_i, n_i\}, R} \left\{ \omega V_E(R) + \int_i [u(c_i) - v(n_i) + \beta \phi R \cdot (a_i + z\theta_i n_i - c_i)] \right\} \quad (4.1)$$

$$\text{s.t. } c_i \leq a_i + \bar{b}(R) \quad (4.2)$$

$$I(R) = \int_i (a_i + z\theta_i n_i - c_i) \quad (4.3)$$

where $V_E(R) = \int_R^\infty I(R) dR$ is the consumption of entrepreneurs in period 2, and ω is the planner's welfare weight on entrepreneurs.

The choice of ω is important because the planner may be tempted to act as a monopolist on behalf of some of the agents, for instance by restricting the supply of funds to drive up the interest rate. I set $\omega = \beta \phi$, so that the planner places equal marginal value

on consumption by entrepreneurs and households in period 2, and is not influenced by this motive.

The first-order conditions of the constrained planner's problem are

$$u'(c_i) = \beta R \phi + \mu_i - \rho \quad (4.4)$$

$$v'(n_i) = z\theta_i \cdot (\beta R \phi - \rho) \quad (4.5)$$

$$\int_i \mu_i \cdot \bar{b}'(R) = -\rho I'(R) \quad (4.6)$$

where ρ is the multiplier on constraint (4.3). Together with complementary slackness conditions on μ_i , these expressions define the constrained optimum allocation.

Since $I'(R) < 0$, $\bar{b}'(R) > 0$, and $\mu_i \geq 0$, we have $\rho \geq 0$, and as long as $\mu_i > 0$ for some i , we have $\rho > 0$. Intuitively, as long as there are constrained households, the planner would like to raise the interest rate, because this would raise \bar{b} . However, increasing R will decrease investment demand, leading to excess savings. Thus the planner can only increase R by raising c or decreasing n for some households.

We can interpret this graphically by making use of our AD-AS diagram from section 3.1. For a given $\rho > 0$, the AD curve shifts out wherever some households are unconstrained, and the AS curve shifts in. Both shifts will tend to drive up the interest rate relaxing the constraint, but the net effect on output is ambiguous.

Intuitively, we can think of ρ as capturing a *pecuniary externality* operating through the interest rate. When households choose consumption and labor, they do not consider the effect of their decisions on the equilibrium interest rate. A higher interest rate will raise the borrowing limit for all households, producing total benefits $\int_i \mu_i \cdot \bar{b}'(R)$. Since households do not internalize these effects, in equilibrium there will be **underconsumption** and **overproduction** relative to the constrained optimum.

It is important to note that overproduction refers to individual household production decisions, not aggregate production. We already saw in section 3.1 that there can be a paradox of thrift, in which decreasing household labor supply at any R leads to higher equilibrium labor supply. Even if the equilibrium is not on the upward sloping part of the aggregate demand curve, the constrained optimum may still have higher equilibrium output than the equilibrium because of higher consumption by unconstrained households.

The following proposition summarizes the main results concerning the constrained optimal allocation.

Proposition 7 (Period 1 Constrained Optimum). *There exists a constrained optimal allocation at which:*

- (i) No borrowing constraint binds iff no borrowing constraint binds in equilibrium.
- (ii) R is strictly greater than in equilibrium if the constraint binds for any household.
- (iii) If the constraint binds for some households, either total consumption is greater than in equilibrium, or the variance of consumption across households is lower.

Proof. See Appendix 1. □

4.1.2 Period 0

Next we turn to period 0. There are two sorts of planners we consider: a planner that chooses both period 0 and period 1 household decisions, who I call the *integrated planner*, and a planner that can only choose period 0 consumption, who I call the *ex-ante planner*.

I find that the integrated planner will not change period 0 decisions from what households would choose when facing the same period 2 consumption. This means that ex-post interventions are sufficient if they can be implemented. By contrast, an ex-ante planner will increase period 0 consumption of wealthy households relative to equilibrium, driving up period 1 demand. Thus in equilibrium households engage in **oversmoothing**, with households with low endowments borrowing too much from households with high endowments.

Integrated Planner We can write the integrated planner's problem as:

$$V_0^p(\{y_{i0}\}) = \max_{\{c_{i0}\}, R_1} \left\{ \omega_0 V_E^0(R_1) + \int_i u(c_{i0}) + \beta E_0 [V_1^p(\{a_i\})] \right\} \quad (4.7)$$

$$\text{s.t. } a_i = R_1(y_{i0} - c_{i0}) \quad (4.8)$$

$$I_0(R_1) = \int_i (y_{i0} - c_{i0}) \quad (4.9)$$

Applying the envelope condition to (4.1), we find $\partial V_1^p / \partial a_i = u'(c_{i1})$. Then the first order conditions of (4.7) are

$$u'(c_{i0}) = \beta R_1 E_0[u'(c_{i1})] - \gamma \quad (4.10)$$

$$-\gamma I_0'(R_1) = \beta \int_i (y_{i0} - c_{i0}) E_0[u'(c_{i1})] - \omega_0 I_0(R_1) \quad (4.11)$$

where γ is the constraint on (4.11).

Since $\int_i (y_{i0} - c_{i0}) = I_0(R_1)$, we may write the second expression as

$$-\gamma I_0'(R_1) = \int_i (y_{i0} - c_{i0}) (\beta E_0[u'(c_{i1})] - \omega_0)$$

which says that γ is positive iff the planner perceives a higher marginal utility of consumption for households than entrepreneurs in period 1, weighted by savings. A positive γ causes the planner to choose higher consumption for households in period 0, acting as a monopolist to reduce savings and drive up the period 1 interest rate. This represents a transfer from entrepreneurs to households in period 1.

If $u'(c_{i1})$ were constant across all states, we could simply choose to weight period 1 consumption by households and entrepreneurs equally, and let $\omega_0 = \beta u'(c_{i1})$. We would then have $\gamma = 0$, and the same household Euler equation as in the decentralized equilibrium. However, if the constraint is binding for some households in some states, marginal utility will not in general be constant across households. Then there is no simple defensible choice of ω_0 that eliminates this mechanism, and the planner will in general make use of it.

Ex-ante Planner Now consider the problem of the ex-ante planner, who may only choose period 1 household consumption.

$$V_0^p(\{y_{i0}\}) = \max_{\{c_{i0}\}, R_1} \left\{ \omega_0 V_E^0(R_1) + \int_i u(c_{i0}) + \beta E_0 [V_1^{eq}(\{a_i\})] \right\} \quad (4.12)$$

$$\text{s.t. } a_i = R_1(y_{i0} - c_{i0}) \quad (4.13)$$

$$I_0(R_1) = \int_i (y_{i0} - c_{i0}) \quad (4.14)$$

where V_1^{eq} is the value function of the planner over the equilibrium, that is

$$V_1^{eq} = \max_{R_2} \left\{ \omega_1 V_1^e + \int_i [u(c_{i1}) - v(n_i) + \beta \phi R_2 \cdot (a_{i1} + z_1 \theta_i n_i - c_{i1})] \right\}$$

$$\text{s.t. } I_1(R_2) = \int_i (a_{i1} + z \theta_i n_i - c_{i1})$$

where c_{i1} and n_i are defined by household decision rules.

Note that although V_1^{eq} is written as a maximization problem in R_2 , for given a_{i1} its value is fully determined. This formulation simply allows the planner to consider the effects of its decisions on the period 2 interest.

The period 0 Euler equation is

$$u'(c_{i0}) = \beta R_1 E_0 [u'(c_{i1}) - \rho \cdot \mathbb{I}_{\mu=0}] - \gamma$$

where ρ and γ are the marginal values of increasing R_1 and R_2 respectively. γ satisfies a similar expression to the integrated planner case, but with the addition of the term

$-\rho \cdot \mathbb{I}_{\mu=0}$:

$$-\gamma I'_0(R_1) = \beta \int_i (y_{i0} - c_{i0}) E_0[u'(c_{i1}) - \rho \cdot \mathbb{I}_{\mu=0}] - \omega_0 I_0(R_1)$$

and ρ gives the value of relaxing the borrowing constraints of constrained households due to the increase in the interest rate, which is given by:

$$\bar{b}'(R_2) \int_i \mu_i = \rho \left(-I'_1(R_2) + \int_i s'_i(R_2) \right)$$

Once more, we can interpret ρ as the marginal value of increasing aggregate demand. Comparing the planner's period 0 Euler equation to the equilibrium expression, the planner chooses higher period 0 consumption in proportion to a household's likelihood of *not* facing binding borrowing constraints in period 1. Since the probability of constraints binding varies between households only by initial assets a_1 , the planner is effectively choosing higher period 0 consumption for relatively wealthy households.

Intuitively, a wider asset distribution entering period 1 increases the probability of binding constraints. The planner internalizes this effect, and adjusts period 0 behavior to narrow the asset distribution by increasing consumption of wealthy households (so that their saving is reduced), and decreasing consumption of poor households (so that their saving is increased). Thus relative to the choice of a constrained planner, the equilibrium exhibits **oversmoothing** of period 0 endowment shocks, where oversmoothing entails *oversaving* by wealthy households, and *overborrowing* by poor households.

Note that the inefficiency does not arise from aggregate overborrowing, but from oversmoothing: overborrowing by poor households *and* oversaving by wealthy households. This is because aggregate assets entering period 1 do not matter for the risk of crisis, but only the asset *distribution*. Thus this result is related to but distinct from the equilibrium overborrowing found in many models of financial amplification, as in e.g. [Lorenzoni \(2008\)](#).

4.2 Policy

(Note: proofs in this section are incomplete)

Next we consider conventional government policies can improve matters. It is natural to divide the discussion between period 1 and period 0. This corresponds to the common distinction between ex post policies undertaken to mitigate an ongoing crisis, and ex ante policies taken to reduce the severity of future crises.

Equilibrium with Government Suppose the government chooses spending G_t in each period $t = 0, 1, 2$. The government issues bonds in periods 0 and 1, which repay at rates R_1 and R_2 at the beginning of the following period. The government also levies a labor income tax in period 1 τ_1^n , and a tax on savings in each period τ_0^s and τ_1^s . Finally, the government levies a lump sum tax T_2 at the end of period 2.

The government's period budget constraints are

$$\begin{aligned} G_0 &= B_0 + \tau_0^s S_0 \\ G_1 &= B_1 - R_1 B_0 + \tau_1^s S_1 + \tau_1^n Y_1 \\ G_2 &= T_2 - R_2 B_1 \end{aligned}$$

The household problem is as in section 2 except for the addition of taxes. Household i faces period budget constraints

$$\begin{aligned} a_{i1} &= R_1(1 - \tau_0^s)s_{i0} = R_1(1 - \tau_0^s)(y_{i0} - c_{i0}) \\ a_{i2} &= R_2(1 - \tau_1^s)s_{i1} = R_2(1 - \tau_1^s)(a_{i1} + (1 - \tau_1^n)z\theta_i n_i - c_{i1}) \\ W_i &= a_{i2} + y_{i2} - c_{i2} - T_2 \end{aligned}$$

together with the constraint on period 1 purchases

$$c_{i1} \leq a_{i1} + \bar{b}_1(R_2)$$

Household optimality conditions in periods 0 and 1 become

$$u'(c_{i0}) = \beta R_1(1 - \tau_0^s)E_0[u'(c_{i1})] \quad (4.15)$$

$$u'(c_{i1}) = \beta R_2(1 - \tau_1^s)E_1[u'(c_{i2})] + \mu_i \quad (4.16)$$

$$v'(n_i) = z\theta_i \beta R_2(1 - \tau_1^n)(1 - \tau_1^s)E_1[u'(c_{i2})] \quad (4.17)$$

Period 2 optimality conditions remain unchanged. If the period 2 constraint never binds, we have $u'(c_{i2}) = \phi$, which simplifies the above considerably. We shall assume that this holds in the rest of this section.

To close the model, we use market clearing in the asset market. Now investment competes with government funds, and so the period asset market clearing conditions

become

$$(1 - \tau_0^s)S_{i0} = (1 - \tau_0^s) \int_i s_{i0} = I_0(R_1) + B_0 \quad (4.18)$$

$$(1 - \tau_1^s)S_{i1} = (1 - \tau_1^s) \int_i s_{i1} = I_1(R_2) + B_1 \quad (4.19)$$

By Walras' Law, we can equivalently write market clearing in the goods market:

$$Y_0 = C_0 + I_0 + G_0 \quad (4.20)$$

$$Y_1 = C_1 + I_1 - A_1 + R_1 B_0 + G_1 \quad (4.21)$$

Borrowing Constraint. The presence of taxes on labor and savings (τ_1^n, τ_1^s) alters the calculation of the borrowing constraint $\bar{b}(R_2)$. Higher taxes on labor or savings reduce the benefit of repaying, and so decrease the borrowing constraint.

Assuming no period 2 constraint, the period 1 borrowing constraint satisfies

$$\bar{b} = (1 - \tau_1^n)z\theta n^* - \frac{v(n^*)}{\beta R_2(1 - \tau_1^s)\phi}$$

where n^* satisfies

$$v'(n^*) = (1 - \tau_1^n)(1 - \tau_1^s)\beta R_2 z\theta \phi$$

The borrowing constraint varies with changes in policies according to:

$$\frac{d\bar{b}}{d\tau_1^s} = -\frac{(1 - \tau_1^n)z\theta n^* - \bar{b}}{1 - \tau_1^s}$$

$$\frac{d\bar{b}}{d\tau_1^n} = -z\theta n^*$$

and varies with changes in parameters according to:

$$\frac{d\bar{b}}{dR_2} = \frac{(1 - \tau_1^n)z\theta n^* - \bar{b}}{R_2}$$

$$\frac{d\bar{b}}{dz} = -(1 - \tau_1^n)\theta n^*$$

$$\frac{d\bar{b}}{d\theta} = -(1 - \tau_1^n)zn^*$$

Fiscal Policy. Governments have historically undertaken fiscal stimulus to raise aggregate demand during crises. In this model, a debt-financed increase in government spend-

ing in period 1 simply shifts the aggregate demand curve to the right by ΔG . This raises the interest rate, loosening borrowing constraints on households, and raises equilibrium output as discussed in section 3.1.

When might such a policy be optimal? To investigate this question, we focus on the case with no income or savings taxes. Then government budget constraints become $B_1 = G_1$ and $T_2 = \beta R_2 B_1$. Suppose that government spending increases household utility by $h(G)$, where $h(\cdot)$ is strictly increasing, strictly concave, and satisfies the Inada conditions.

Consider the problem of a planner who chooses G_1 , taking household equilibrium decision rules as given. The optimal policy problem is

$$\max_{R_2, B_1, G_1} \left\{ \omega V_E(R_2) + h(G_1) + \int_i [u(c_i) - v(n_i) + \beta R\phi \cdot s_i - \beta\phi R B_1] \right\} \quad (4.22)$$

$$\text{s.t. } B_1 + I(R) = \int_i (a_i + z\theta_i n_i - c_i) \quad (4.23)$$

$$B_0 R_1 + G_1 = B_1 \quad (4.24)$$

where $s_i = a_i + z\theta_i n_i - c_i$. This yields optimality condition

$$h'(G) + \rho = \beta R\phi \quad (4.25)$$

where ρ is the Lagrange multiplier on (4.23). The lefthand side of (4.25) is the marginal value of increasing G , which is composed of the direct utility of government spending $h'(G)$, and the value of increasing aggregate demand ρ , which satisfies

$$\rho = \frac{1}{S'(R) - I'(R)} \int_i \mu_i \frac{d\bar{b}}{dR}$$

The term $1/(Y'(R) - D'(R))$ is $\partial R/\partial G$, while $\int_i \mu_i \frac{d\bar{b}}{dR}$ is the value of increasing R . Compare with the government spending multiplier derived in section 3.1

$$\frac{dY}{dG} = \frac{Y'(R)}{Y'(R) - D'(R)}$$

which is just $\partial R/\partial G \cdot Y'(R)$.

Denote optimal government spending by G^* . Since $h(\cdot)$ is strictly concave, (4.25) implies that G^* is decreasing in R and increasing in ρ . Following the terminology used in Werning (2011), the former is the *opportunistic motive*, whereby low interest rates prompt higher government spending to take advantage of cheaper financing, and the latter is the *stimulus motive*, since when constraints are binding increased government spending raises

the interest rate and relaxes these constraints.

We can formalize these notions in the following proposition:

Proposition 8. *In an economy with no binding constraints in the initial equilibrium,*

- (i) *A decrease in aggregate demand will cause optimal G to rise.*
- (ii) *Any shock that causes constraints to bind and does not cause R_2 to rise will cause optimal G to rise.*

Proof. See Appendix 1. □

A common way of quantifying the effects of fiscal policy is by means of the government spending multiplier. Here the spending multiplier is $\frac{Y'(R)}{Y'-D'(R)}$, which we can write as $\frac{1}{1-\frac{D'(R)}{Y'(R)}}$. This is similar to the traditional expression $\frac{1}{1-mpc}$, where mpc is the marginal propensity to consume. This formulation can be interpreted as the sum of a geometric series, i.e. an exogenous increase in demand X prompts a further increase $mpc \cdot X$, which in turn prompts a further increase $mpc^2 \cdot X$, etc.

Here an increase in demand X will prompt an increase in R of $1/Y'(R) \cdot X$, the quantity necessary to stimulate additional production to meet this demand. If sufficiently many households are borrowing constrained, this increase in R will raise demand by $\frac{D'(R)}{Y'(R)}X$, and so the geometric sum proceeds analogously to the case with fixed mpc .

It is clear from the expression above that the marginal multiplier on government spending is greater than 1 if and only if the aggregate demand curve is upward sloping, i.e. $D'(R) > 0$. A necessary but not sufficient condition for this to prevail is that aggregate consumption demand be increasing in R , which will only occur when a relatively high fraction of households are constrained. However, it is important to note that the stimulus motive for government spending holds as long as $\rho > 0$, i.e. as long as constraints are binding on *any* household, even if the multiplier is less than 1.

Period 0 Equilibrium Now consider changes in period 0 policies. In characterizing the period 0 equilibrium, it is most convenient to focus on the asset market equilibrium. Market clearing requires

$$(1 - \tau_0^s)S(R_1) = I_0(R_1) + B_0$$

A change in government spending has no direct effect in this market. Indirectly, the increased spending must be financed by either new bond issuance B_0 or taxes on saving τ_0^s . The former will increase demand for loanable funds, shifting the demand curve to the right, while the latter will reduce the supply of savings, shifting the asset supply curve to the left. We consider these each in turn.

First consider new bond issuance B_0 . This results in a rightward shift in the loanable funds demand curve, which will increase equilibrium interest rates R_0 and savings S_0 .

This bond issuance will also affect period 1 equilibrium by changing the distribution of assets a_1 entering period 1. Since both S_0 and R_0 increase, total assets A_1 increase. However, since B_0R_1 also enters into aggregate demand, the effect on period 1 AD is ambiguous. In particular, the term $B_0R_1 - A_1$ enters into AD. Letting $A_1 = S_0R_1$, we can write this as $-(S_0 - B_0)R_1$, or equivalently $I_0(R_1) \cdot R_1$. This will increase in R_1 iff period 0 investment is elastic, i.e.

$$\left\| \frac{I'_0(R_1) \cdot R_1}{I_0(R_1)} \right\| > 1$$

A similar effect occurs if the government taxes savings directly. In this case, aggregate savings fall, and the interest rate R_1 rises. Once more there is a change in A_1 , though there is now no change in B_1 . The net effect on aggregate demand depends on whether the fall in S_0 or the rise in R_1 is larger. This time, an increase in τ_0^s will tend to decrease $S_0 - B_0 = I_0$, and so, holding $C_1 + I_1$ fixed, will *decrease* AD iff period 0 investment is elastic.

Period 0 bond issuance and taxes on savings will also affect the distribution of period 0 assets. However, the effect is difficult to characterize in general, even ignoring general equilibrium effects. At best we can sketch a rough argument based on the following lemma:

Lemma 5. *Given two households with the same consumption in periods 0 and 1, one of whom is constrained in period 2 and the other unconstrained, the following relations hold:*

$$\begin{aligned} \left(\frac{dc_{i0}}{d\tau_0^s} \right)^{unc} &> \left(\frac{dc_{i0}}{d\tau_0^s} \right)^{con} > 0 \\ \left(-\frac{dc_{i0}}{dR_1} \right)^{con} &< \left(-\frac{dc_{i0}}{dR_1} \right)^{unc} > 0 \end{aligned}$$

Proof. See Appendix 1. □

Lemma 5 suggests that an increase in B_0 resulting in an higher interest rates R_1 will decrease demand for consumption from wealthy households by more than poor households, since poor households are more likely to be constrained in period 1. Thus savings rise by more for rich households, who were saving more to begin with, which will likely increase the variance of savings.

5 Conclusion

I have developed a model of an amplification process operating through the consumer credit. Binding constraints on consumers decrease demand, resulting in inefficiently low production and low real interest rates.

The value of borrowing constraints depend on the price of current output, i.e. the real interest rate. Any fall in aggregate demand will lower the real interest rate, tightening borrowing constraints and reducing demand further. Therefore binding borrowing constraints amplify demand shocks. By contrast, a positive supply shocks lowers the real interest rate, tightening constraints and offsetting the effect of the supply shock, either partially or entirely.

The probability of binding constraints depends on the distribution of initial assets. When the distribution is widened, so that some households have more and some fewer assets, the probability of binding constraints increases, and total aggregate demand falls.

Relative to the choices of a constrained planner, equilibrium when constraints are binding exhibits overproduction and underconsumption. The ex-ante equilibrium when constraints are loose exhibits oversmoothing: overborrowing by poor households, and oversaving by rich households.

I examined optimal government policy, and found that government spending is increasing when constraints are binding for *two* reasons. Since the real interest rate is lower when borrowing constraints are binding, financing government spending is cheaper and so optimal spending increases due to the opportunistic motive. Since increased spending raises aggregate demand and relaxes the borrowing constraint, government spending is also higher due to the stimulus motive. While the government spending multiplier may be greater than one when constraints are binding for a large fraction of households, this is not a necessary condition for the stimulus motive to be in effect. As long as borrowing constraints are binding on any households, optimal government spending is higher.

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Appendix 1: Omitted Proofs

Proofs from Section 2.1

Proof of Lemma 1. To see that $G(\cdot)$ does not depend on a_1 , note that \bar{b}_1 only appears in V^d and V^* in the term $u(a_1) + \bar{b}_1$, which drops out when we subtract V^d from V^* .

To see that $G(\cdot)$ is strictly decreasing in \bar{b}_1 , we differentiate V^d and V^* with respect to \bar{b}_1 , and find that $G'(\bar{b}_1) = \mu_i^R - u'(a + \bar{b}_1)$. Clearly this is negative when the constraint is not binding, i.e. $\mu_i = 0$, because households in period 0 will always choose a_1 such that $a_1 + \bar{b}_1 > 0$, because $u'(c) \rightarrow \infty$ as $c \rightarrow 0$. When the constraint binds, $\mu_i = u'(a + \bar{b}_1) - \beta R_2 E_1[u'(c_2)]$, and so $G'(\bar{b}_1) = -\beta R_2 E_1[u'(c_2)]$.

To see that $G(\cdot)$ is increasing in both z and θ , we can simply note that V^R is increasing in both, and neither appear in the statement of V^d . \square

Proofs from Section 3.1

Proof of Lemma 2. From (2.19), we have $\bar{b}'(R) = (z\theta n - \bar{b})/R_2$, and from (2.18) we have $\bar{b} = z\theta n - v(n)/(\beta\phi R)$, when the period 2 constraint does not bind (so that $\Delta^* = \Delta^d$). Combining these, we find $\bar{b}'(R) = v(n)/(\beta\phi R^2)$, which is strictly positive. \square

Proof of Lemma 3. Recall that unconstrained consumption c^u satisfies $u'(c^u) = \beta\phi R$. Then as $R \rightarrow 0$, $u'(c^u) \rightarrow 0$, which happens only when $c^u \rightarrow \infty$.

Next, consider the definition of \bar{b} in (2.18):

$$\bar{b} = \frac{\beta\phi R z \theta n - v(n)}{\beta\phi R}$$

Since n is defined by $v'(n) = z\theta R\beta\phi R$, $n \rightarrow 0$ as $R \rightarrow 0$. Therefore both the numerator and denominator of this expression go to 0. Using l'Hôpital's Rule

$$\begin{aligned} \lim_{R \rightarrow 0} \frac{d\bar{b}}{dR} &= \lim_{R \rightarrow 0} \frac{\frac{d}{dR} (\beta\phi R z \theta n - v(n))}{\frac{d}{dR} (\beta\phi R)} \\ &= \lim_{R \rightarrow 0} \frac{\beta\phi z \theta n + [\beta\phi R z \theta - v'(n)] \frac{dn}{dR}}{\beta\phi} \end{aligned}$$

Observe that $\beta\phi R z \theta - v'(n) = 0$ from the labor optimality condition. Therefore

$$\lim_{R \rightarrow 0} \frac{d\bar{b}}{dR} = \lim_{R \rightarrow 0} (z\theta n) = 0$$

Finally, since $a_{i1} < \infty$ for all households, the combination of $c^u \rightarrow \infty$ and $\bar{b} \rightarrow 0$ implies that all households become constrained as $R \rightarrow 0$. \square

Proof of Corollary 1. The slope of $C(R)$ is equal to $\int_i c'_i(R)$ over all households. If a household is unconstrained, we have $c'_i(R) = \beta\phi/[u''(c)]$, which is strictly negative by assumption. Therefore $c'_i(R)$ is strictly greater for constrained households than for unconstrained, and so the integral over all households is strictly greater when some households are constrained than when all households are unconstrained.

For the second part, observe that as $R \rightarrow 0$, the constraint binds for all households (as shown in Lemma 3), and so there exists some range $[0, \hat{R}]$ in which all households are constrained. Thus in this range, $C(R)$ is increasing in R . \square

Proof of Proposition 1. Consider the function $g(R) = a + \bar{b}(R) - c^u(R)$. Now I claim that as $R \rightarrow \infty$, $g(R) \rightarrow \infty$. This is because c^u satisfies $u'(c^u) = \beta R\phi$, $\beta\phi > 0$, and by assumption $u'(c) \rightarrow \infty$ as $c \rightarrow 0$. Likewise, as $R \rightarrow \infty$, we have $n \rightarrow \infty$ since $v'(n) = z\theta\beta R\phi$. We have $\bar{b} = (z\theta n\beta\phi R - v(n))/(\beta\phi R)$, and as $R \rightarrow \infty$, both the numerator and denominator go to 0. Using l'Hôpital's rule and differentiating with respect to R , we obtain

$$\lim_{R \rightarrow \infty} \bar{b}(R) = \frac{(z\theta\beta\phi R - v'(n)) \cdot \frac{dn}{dR} + z\theta\beta\phi n}{\beta\phi} = z\theta n$$

which goes to ∞ as $R \rightarrow \infty$.

By IVT, c^u and $a + b(R)$ have a unique intersection.

Then by the intermediate value theorem, $g(R)$ must intersect 0 for some $R \in (0, \infty)$. Moreover, since $g(R)$ is strictly decreasing, this intersection must be unique.

Call this intersection \bar{R}_i . Then we observe that because $g(R)$ is strictly decreasing, for $R < \bar{R}_i$ we have $a + \bar{b} > c^u$, meaning the constraint does not bind, and for $R > \bar{R}_i$ we have $a + \bar{b} < c^u$, meaning the constraint binds. \square

Proof of Proposition 2. Let R_1 be the interest rate at the intersection of $Y(R)$ and $D_1(R)$, and R_2 the interest at the intersection of $Y(R)$ and $D_2(R)$. Since the intersections are unique, for all $R > R_2$ we have $Y(R) > D_2(R)$. Since $D_2(R) \geq D_1(R)$, this likewise implies $Y(R) > D_1(R)$, and so $R_1 \leq R_2$. Since $Y(R)$ is strictly increasing, this likewise implies that $Y(R_1) \leq Y(R_2)$. Further, if $D_1(R_2) < D_2(R_2)$, then $R_1 \neq R_2$, and so $R_1 < R_2$ and $Y(R_1) < Y(R_2)$. \square

Proof of Corollary 2. For any given (θ, z, a) , let $Y(R)$ be the aggregate supply curve and $D^u(R)$ the unconstrained aggregate demand curve. Now observe that because $Y(R)$ is strictly increasing in R , whereas $D^u(R)$ is strictly decreasing in R , and as $R \rightarrow 0$ we have $C^u(R) \rightarrow \infty$ and $Y(R) \rightarrow 0$, and as $R \rightarrow \infty$ we have $C^u(R) \rightarrow 0$ and $Y(R) \rightarrow 0$, excess supply $Y(R) - D^u(R)$ is strictly increasing in R and attains a unique intersection at $R^u \in (0, \infty)$. This is the unconstrained equilibrium.

Now let $D^c(R)$ be the constrained aggregate demand curve. Note that the constrained AD curve satisfies $D^c(R) \leq D^u(R)$, with strict inequality iff at least one constraint is binding. If $D^c(R) = D^u(R)$ at the R^u , then the constrained equilibrium is the same as the unconstrained equilibrium, and no borrowing constraints are binding.

Suppose instead $D^c(R) < D^u(R)$. Then we argue that constrained equilibrium output is lower than unconstrained output. Consider R^c that is a constrained equilibrium, i.e. $D^c(R^c) = Y(R^c)$. At this point, the constraint must be binding because otherwise this would be the unconstrained equilibrium. Therefore we have $D^u(R^c) > D^c(R^c) = Y(R^c)$. Therefore $Y(R^c) - D^u(R^c) < 0$, and since $Y(R) - D^u(R)$ is strictly increasing in R , and is equal to 0 at R^u , this implies that $R^c < R^u$. Since $Y(R)$ is strictly increasing in R , this implies $Y(R^c) < Y(R^u)$. \square

Proof of Proposition 3. Consider a shift in the variable x . Then the change in output will be equal to

$$\frac{dY}{dx} = \frac{\partial Y}{\partial x} + \underbrace{\frac{\partial Y}{\partial R} \cdot \frac{dR}{dx}}_{\text{GE Effects}}$$

We can calculate dR/dx by applying the implicit function theorem to $Y(R) = D(R)$, i.e.

$$\frac{dR}{dx} = \frac{\frac{\partial Y}{\partial x} - \frac{\partial D}{\partial x}}{\frac{\partial D}{\partial R} - \frac{\partial Y}{\partial R}}$$

Combining and rearranging yields the expression above.

Now consider the effect of a decrease in $\partial D/\partial R$ on dY/dx , leaving all other terms unchanged. Simple differentiation yields

$$\frac{\partial \left(\frac{dY}{dx} \right)}{\partial \left(\frac{\partial D}{\partial R} \right)} = \frac{\left(\frac{\partial D}{\partial x} - \frac{\partial Y}{\partial x} \right) \frac{\partial Y}{\partial R}}{\left(\frac{\partial D}{\partial R} - \frac{\partial Y}{\partial R} \right)^2}$$

which is positive iff $\frac{\partial D}{\partial x} > \frac{\partial Y}{\partial x}$. \square

Proofs from Section 3.2

Proof of Lemma 4. For (i), we need to show that \bar{b} increases when \bar{m} decreases. To do so, we use the expression

$$\bar{b} = z\theta n - v(n)/(\beta\phi R) + \frac{\Delta^* - \Delta^d}{\phi R}$$

First observe that $\Delta^d = u(\bar{m}) - \phi\bar{m}$, because of the assumption that $u'(\bar{m}) > \phi$. Therefore $-d\Delta^d/d\bar{m} = \phi - u'(\bar{m}) < 0$.

First suppose that $u'(\bar{m} + z\theta n - \bar{b}) < \phi$, so that the period 2 constraint does not bind for the (*) case. Then n^* will also not change, and so the only effect on \bar{b} from a fall in \bar{m} will be the fall in Δ^d , so that \bar{b} will necessarily increase. **(Note: This proof is incomplete.)**

Next suppose that $u'(\bar{m} + z\theta n - \bar{b}) > \phi$, so that the period 2 constraint binds. First suppose that we hold n^* unchanged. Then the only effect of the decrease in \bar{m} will be to decrease Δ^d and Δ^* . The marginal change in Δ^* will be $\phi - u'(c_2^*)$, and the net change in the expression for \bar{b} (holding n^* fixed) will be $u'(c_2^d) - u'(c_2^*)$. Then assuming $z\theta n > \bar{b}$, we have $c_2^d < c_2^*$, and so $u'(c_2^d) > u'(c_2^*)$, so that $\Delta^* - \Delta^d$ rises.

Now suppose that we allowed n^* to change as well. Recall that n^* was chosen to maximize

$$u(a + \bar{b}) - v(n) + \beta [u(c_2) - \phi c_2 + \phi R (z\theta n - \bar{b})]$$

Observe that by removing and adding constant terms unaffected by n , this is the same as maximizing

$$\beta\phi R \left(z\theta n - \frac{v(n)}{\beta\phi R} + \frac{\Delta^* - \Delta^d}{\phi R} \right)$$

Therefore adjusting n^* in response to the change in \bar{m} can only increase this term further, so that \bar{b} will necessarily rise.

It remains only to show that $\bar{b} < z\theta n$. Suppose instead we have $z\theta n < \bar{b}$. Then we would have $\Delta^* \leq \Delta^d$, with strict inequality when the period 2 constraint was binding for the (*) case. Observe that we can rewrite the expression for \bar{b} as

$$z\theta n - \bar{b} = \frac{v(n)}{\beta R \phi} - \frac{\Delta^* - \Delta^d}{\phi R}$$

Then since $v(n) > 0$ and $\Delta^* - \Delta^d < 0$, we have $z\theta n - \bar{b} > 0$, contradicting our assumption.

For (ii), first suppose there is no change in c_1 or n in response to the fall in \bar{m} . Since the constraint is binding, we have $c_2 = sR + \bar{m}$, where $s = a + z\theta n - c$. Thus the fall in \bar{m} mechanically leads to a fall in c_2 , and so an increase in $u'(c_2)$.

Now suppose there is a response in n or c_1 . To reverse the rise in λ , these responses

must be either a rise in n or a fall in c_2 . But optimal n satisfies $v'(n) = z\theta\beta R(\phi + \lambda)$, and so n will only rise if λ rises. Likewise, unconstrained c_1 satisfies $u'(c_1) = \beta R(\phi + \lambda)$, and so c will only fall if λ rises, while if the household is constrained in period 1, it will increase c_1 due to the rise in \bar{b} . Thus in each case, λ must rise.

For (iii), it is sufficient to show that no household becomes unconstrained following a fall in \bar{m} . Equivalently, no household with $\lambda > 0$, finds itself with $\lambda = 0$ after a decrease in \bar{m} . This follows immediately from (ii), which showed that λ can only rise in response to a fall in \bar{m} . \square

Proof of Proposition 4. As we showed in Lemma 4, decreasing \bar{m} has two marginal effects: increasing λ for unconstrained households, and increasing \bar{b} .

For (i), note that neither of these affect unconstrained households.

For (ii), we observe that $v_n = z\theta\beta R(\phi + \lambda)$, and so an increase in λ will lead to an increase in n .

For (iii), we observe that unconstrained consumption demand satisfies $u_c = \beta R(\phi + \lambda)$, and so an increase in λ results in an increase in c .

For (iv), we consider the effect of an increase in λ . Using comparative statics on the FOCs, we obtain $dc/d\lambda = \beta R/u''(c)$ and $dn/d\lambda = z\theta\beta R/v''(n)$. Substituting $\sigma = -cu''/u'$ and $\xi = nv''/v'$, we obtain $dc/d\lambda = -\beta R c/(\sigma u')$, and $dn/d\lambda = z\theta\beta R n/(\xi v')$. Next we use $v' = z\theta u'$ to obtain $dc/d\lambda = z\theta\beta R c/(\sigma v')$. Next observe that the change in AS minus the change in AD is

$$z\theta dn/d\lambda - dc/d\lambda = z\theta\beta R/u' \cdot (z\theta n/\xi - c/\sigma)$$

which is positive iff the given condition holds. \square

Proofs from Section 3.3

Proof of Proposition 5. Since R_1 and R_2 are fixed, the unconstrained level of consumption in periods 0 and 1 are fixed. Therefore an unconstrained household will consume none of an increase in endowment. By contrast, a constrained household will consume a fraction

$$\frac{dc_0}{dy_0} = \frac{\beta R_1^2 u''(c_1)}{u''(c_0) + \beta R_1^2 u''(c_1)} \in (0, 1)$$

of an increase in endowment, and save the rest. Moreover, since a household is constrained only if it saves less than $\bar{s} = (c^u - \bar{b})/R_1$, unconstrained households save strictly more than constrained households.

Now consider a transfer of dt from a mass m of unconstrained households to a mass m of constrained households. Suppose for simplicity that the households are identical, the constrained households initially save $s_H \geq \bar{s}$, and unconstrained households save $s_L < \bar{s}$. As shown above, the marginal propensity to save for the unconstrained households is 1, and the marginal propensity to save for constrained households is $\gamma \in (0, 1)$.

Total savings are equal to $S = \int_i s_i$, and as a result of the transfer dt we find that total savings increase by $m(1 - \gamma)dt$. Therefore the transfer increases total desired savings.

The variance of desired savings is $\sigma_s^2 = \int_i (s_i - S)^2$. After the transfer, the variance increases by

$$d\sigma_s^2 = 2 \left(\int_i (s_i - S)(1 - \gamma) - \gamma(s_L - s) + (s_H - s) \right) m dt$$

Since $\int_i s_i = S$, the term under the integral is zero. Thus this term is positive as long as $s_H > \gamma s_L + (1 - \gamma)S$, a sufficient condition for which is $s_H \geq S$. \square

Proof of Proposition 6. Suppose that initially a_{i1} are drawn from the distribution a . Then consider a mean-preserving spread (mps) of a , $a' = a + \epsilon$, where ϵ is a random variable with mean 0 and variance $\sigma^2 > 0$.

For given state (z, θ) the aggregate demand curve satisfies $D(R) = C(R) + I(R)$, where for a given R , $C = \int_i c_i$. Since consumption does not depend on θ_i , we can alternatively integrate over a , so that $C = E[c(a)]$, where $c(a) = \min(a + \bar{b}, c^u)$, where c^u is the unconstrained desired level of consumption, which is the same for all households.

By the law of iterated expectations, we can write $C = E[c(a')] = E_a[E_\epsilon[c(a + \epsilon)|a]]$. Since $c(a)$ is concave in a , by Jensen's inequality we have $C \leq E_a[c(E_\epsilon[a + \epsilon|a])]$. Since a and ϵ are independent, and since $E[\epsilon] = 0$, $E[a + \epsilon|a] = a$, and so we have $C \leq E[c(a)]$.

Since $D(R)$ decreases for all R , the period 1 aggregate demand curve shifts in, which by Proposition 2 decreases R and equilibrium output Y . \square

Proofs from Section 4

Proof of Proposition 7. For the only if part of (i), we must prove that if no constraint is binding in equilibrium, then no constraint is binding in the constrained optimum. Observe that if no constraint binds in equilibrium, $\mu_i = 0$ for all households. Then the equilibrium allocation satisfies all the conditions for the constrained optimum with $\mu_i = 0$ for all i and $\rho = 0$.

For the if part of (i), we must prove that if no constraint is binding in the constrained optimum, then no constraint is binding in equilibrium. Observe that if no constraint were

binding in the constrained optimum, then $\rho = 0$, and all other conditions would be the same as in equilibrium. Therefore the constrained optimal allocation would also be the equilibrium allocation, in which no constraint binds.

For (ii), we first observe that an increase in ρ is equivalent to a outward shift in an AD curve, and an inward shift in an AS curve, with the constrained optimum the intersection of these two curves. From Proposition 2 we know that an outward shift in an AD curve will raise R and Y . Then all that remains is to establish that a leftward shift in an AS curve will raise R at the intersection.

We can write this as $dY/dR < 0$, where dY is the shift in the AS curve. Recall that R is defined by $Y(R) = D(R)$. Then by the implicit function theorem, $dR/dY = [D'(R) - Y'(R)]^{-1}$. Then since $Y'(R) > D'(R)$, we have $dR/dY < 0$.

For (iii), let $G(a)$ be the cdf for the distribution of assets a_1 . Then total consumption is $C = \int_0^\infty c(a)dG(a)$, where $c(a) = \min(a + \bar{b}, c^u)$, with c^u the unconstrained consumption. We can write this as

$$C = \int_0^{\bar{a}} (a + \bar{b})dG(a) + \int_{\bar{a}}^\infty c^u dG(a)$$

where $\bar{a} = c^u - \bar{b}$ the level of assets at which households become unconstrained.

Now observe that at the constrained optimum, R is higher than in the equilibrium, so that \bar{b} is also higher, whereas c^u may be higher or lower. We have $dC/d\bar{b} = \int_0^{\bar{a}} dG(a) > 0$ and $dC/dc^u = \int_{\bar{a}}^\infty dG(a) > 0$. Thus a marginal increase in \bar{b} and c^u will necessarily raise C .

Suppose instead that c^u falls. Then we can show that the variance of c must increase. The variance of c is $\sigma_c^2 = \int_0^\infty [c(a)]^2 dG(a) - C^2$. We can write this as

$$\sigma_c^2 = \int_0^{\bar{a}} (a + \bar{b})^2 dG(a) + \int_{\bar{a}}^\infty c^u dG(a)$$

Then differentiating, we obtain $dV/d\bar{b} = 2 \int_0^{\bar{a}} (a + \bar{b} - C) dG(a)$, and $dV/dc^u = 2 \int_{\bar{a}}^\infty (c^u - C) dG(a)$.

Now observe that we can rearrange the expression for C as

$$\int_0^{\bar{a}} (a + \bar{b} - C) dG(a) + \int_{\bar{a}}^\infty (c^u - C) dG(a) = 0$$

Since consumption can never exceed c^u , average consumption must be less than c^u , and so $c^u - C < 0$. This implies that $\int_0^{\bar{a}} (a + \bar{b} - C) dG(a) < 0$, and thus the variance of c is decreasing in \bar{b} and increasing in c^u . □

Proof of Proposition 8. □

Proof of Lemma 5.

□