Financial Intermediation and the Supply of Liquidity*

Jonathan Kreamer†

August 2017

Abstract

I study the role of financial intermediaries in supplying liquidity to the real economy. Firms hold liquid assets to meet unanticipated expenses. Financial intermediaries supply liquidity by pooling partially liquid assets, but their ability to commit future funds depends on their capital. When liquidity is scarce, there is a positive liquidity premium and investment is inefficiently low. Bank losses raise the liquidity premium and reduce investment. I analyze the optimal supply of public liquidity and find that when private liquidity is scarce the government should issue bonds for their liquidity properties, providing justification for countercyclical budget deficits.

JEL Codes: E43, E44.
Keywords: Liquidity, Financial Intermediation, Financial Crisis.

*I would like to thank Anton Korinek for his advice, encouragement, and many helpful comments. I would like to thank John Shea, Borağan Aruoba, and Pablo D'Erasmo for helpful conversations. I would like to thank participants of the UMD Macro Workshop and the Midwest Macro Meetings for helpful comments.

†Florida State University, Department of Economics
website: http://www.jonathankreamer.com email: jkreamer@fsu.edu

1 Introduction

An important function of financial intermediaries is liquidity provision, i.e. the creation of financial instruments for firms and households to store wealth while maintaining access to these funds to meet unpredictable financing needs. Though liquidity shortages played a prominent role in the 2008 financial crisis, relatively little macroeconomic analysis has focused on the role of the financial sector in providing liquidity to the productive sector. Such analysis is particularly relevant to policy, since government liabilities comprise a significant fraction of the economy's aggregate supply of liquid assets. This paper analyzes the role of financial intermediaries in liquidity provision. I investigate two questions: What effect do fluctuations in the value of intermediaries' balance sheets have on the supply of liquidity to the productive sector? What is the role of public liquidity provision in response to such fluctuations?

To study these questions, I build a model in which investment projects are subject to uncertain ongoing financing needs. Since agency costs limit external financing, firms hold liquid assets to meet these expenses. A portion of these liquid assets are supplied by financial intermediaries, whose ability to issue these assets is linked to the quality of their balance sheets. Under these assumptions, variations in financial sector net worth affect the aggregate supply of liquidity. I find that the aggregate scarcity of liquidity can be summarized by the liquidity premium, i.e. the difference between the price of liquid assets and the price of illiquid assets with the same return. When supply of liquidity is limited, there is a positive equilibrium liquidity premium, and investment falls below its constrained optimal quantity. In such liquidity-constrained states, it is optimal for the government to issue debt for its liquidity properties, and the optimal issuance of public debt is decreasing in bank equity. This provides justification for countercyclical expansion of government balance sheets, as government bonds substitute for private liquidity provision.

As this paper is concerned with the supply of liquidity, I begin by defining what I mean by the term. By liquidity (or equivalently, liquid assets), I mean assets held by firms and households that serve as a store of value, and that can be sold (liquidated) for their full value at any time before maturity. Examples of such assets include government bonds, some corporate bonds, commercial paper, money market funds, and demand deposits issued by banks. The key feature of liquid assets is information insensitivity, and in particular the lack of private information about their value — since all agents know these assets' worth, they can be sold for their full price without discounts due to adverse selection.[1]

I model the demand for liquidity from the business sector in a manner similar to Hölmstrom and Tirole (1998). I suppose that investment projects are subject to stochastic needs for funds. Firms can raise these funds by new borrowing, but due to agency costs are unable to pledge the full value of their projects. If required funds exceed the pledgeable amount, the firm will be borrowing constrained. Anticipating this possibility, firms undertaking a new project can borrow more than required for the initial investment, and hold the excess as liquid assets. Should a need for additional funds arise, firms use these assets as collateral to enable additional borrowing. If liquid assets yield a return equal to the firm's cost of borrowing, firms can hold the optimal amount of liquid assets given their initial borrowing capacity. However, when liquidity is scarce liquid assets command a premium over illiquid assets. Then firms must use some of their limited pledgeable funds to pay this premium, and reduce investment below the constrained efficient level.

Firms obtain these funds in two ways. First, firms may borrow additional funds when they are not borrowing constrained, and use them to purchase assets that can be liquidated in the event of a liquidity shock. This arrangement is equivalent to an insurance contract between firms and original investors, in which investors agree to provide additional financing to firms that require it, even when the firm is unable to raise new financing on private markets. The liquid assets held by firms serve as collateral on this promise, since otherwise investors might not want to provide these funds. Alternatively, firms may provide for their future liquidity needs by purchasing a credit line from a financial intermediary. Typically the firm pays a premium to an intermediary to provide financing at specified terms up to some limit. Note that this is analogous to the arrangement above, except that banks are presumed to be able to credibly provide the promised financing without posting collateral. Thus in equilibrium the premium paid on bank credit lines will equal the liquidity premium on liquid assets.[2]

Financial intermediaries are able to issue credit lines because they are able to commit to supplying future funds at a loss. However, I assume that financial intermediaries are themselves limited by agency costs which depend on their net worth.[3] When banks

[1]The importance of adverse selection in generating market illiquidity was first pointed out by Akerlof (1970). The importance of information insensitivity was emphasized by Gorton and Pennacchi (1990), DeMarzo and Duffie (1999), and recently by Dang et al. (2017). For a summary of the various sources of illiquidity, see Tirole (2011) and Hölmstrom and Tirole (2011).

[2]In fact, credit lines can be considered equivalent to a sort of demand deposit, and the model can be interpreted in this way as well.

[3]Several papers use agency costs in models of banks. One notable example is Mattesini, Monnet, and Wright (2009), which argues that agents with large stakes in the continuation of the economy are better suited to serve as banks. This is analogous to my model, in which bank agency costs depend on their asset holdings.

have low net worth, they have less collateral to be seized in the event of bankruptcy and therefore have more incentive to engage in fraud, for instance by failing to exert effort in screening loan applicants, or by taking on excessively risky loans since they do not bear downside risk. When bank net worth is low, the supply of liquidity will be limited, and the economy will not be able to achieve perfect pooling of idiosyncratic liquidity risk. This provides mechanism for a counteryclical supply of liquidity.

I show that the balance of liquidity in the economy can be summarized by the liquidity premium, i.e. the difference in price between liquid and illiquid assets after adjusting for expected return.[4] When there is a sufficient supply of liquid assets, the liquidity premium is zero, since at the margin liquid assets are held for their expected return. However, when the stock of liquid assets falls below a critical threshold, there is a positive liquidity premium. This reduces equillibrium investment below the constrained optimal level.

Since there are two kinds of liquidity in my model, I can distinguish between two channels of liquidity destruction that occur during a financial crisis. First, a recession may lead to losses on liquid assets directly. For instance, mortgage-backed securities made up a significant fraction of asset-backed commercial paper in 2008, and so the collapse of the housing bubble and resulting increase in morgage delinquincies represented the destruction of a fraction of the economy's stock of liquid assets.[5] Further, a recession will generally involve losses on bank loans which reduce bank net worth, causing banks to reduce credit supply, including liquid instruments such as lines of credit held by firms. I refer to this second mechanism as the *bank channel*.

I next analyze optimal government policy within the context of this model. When there is a positive liquidity premium in equilibrium, the government can improve the allocation by issuing liquid liabilities such as government debt. If issuing bonds is costless, then the optimal policy is to issue bonds until the liquidity premium is zero. However, I consider the case where there is a cost to issuing government debt due to distortionary taxation. I find that when there is a positive liquidity premium at an interior equilibrium, it will always be optimal for the government to provide some public liquidity. This result is quite interesting by itself, as it implies that in an economy with a positive liquidity premium there is positive value of issuing government debt. If total government debt is sufficiently low, the issuance of additional debt will crowd in investment, and these benefits will outweigh the costs of higher debt.

I also analyze how the optimal supply of government debt varies with the supply of

[4]There is ample evidence of positive liquidity premia in reality, with the paradigmatic example being money. However, other liquidity premia have been documented. For example, Krishnamurthy and Vissing-Jorgensen (2012) find that government bonds hold a liquidity premium over corporate bonds.

[5]As documented by Brunnermeier (2009).

private liquidity. I find that when the stock of bank capital falls, the optimal supply of public liquidity increases. This result suggests that the government should issue more debt when the banking sector is poorly capitalized, which is likely true during recessions. This provides an additional justification for procyclical budget deficits, but for completely different reasons than what is usually offered. The benefit arises from the liquidity service provided by government debt, rather than from the aggregate demand effects of higher spending.

By contrast, when the economy's stock of private non-bank liquid assets declines, optimal public liquidity provision *decreases*. This result is driven by the elasticity of private liquidity supply. An increase in government bonds crowds out private liquidity, meaning that government bonds raise total liquidity less than one-for-one. The degree of crowding out depends on the elasticity of private liquidity supply. Since the stock of non-bank liquid assets is taken to be inelastically supplied, a decline in this stock increases the elasticity of private liquidity supply, and crowding out is greater. This decreases the effectiveness of public liquidity provision, leading the government to reduce liquidity supply.

Related Literature. Many papers have examined the role of banks in creating liquid assets. In Diamond and Dybvig (1983), banks offer liquidity insurance by pooling claims to investment projects and selling demand deposits to households. Hölmstrom and Tirole (1998) model liquidity in a framework similar to mine, and show that banks can perfectly insure against idiosyncratic liquidity shocks but not aggregate liquidity shocks. Brunnermeier and Sannikov (2016) introduce intermediaries to a model with public and private liquidity, and show that a fall in bank capital will decrease the supply of inside liquidity.[6] I depart from these models by making bank lending subject to an agency cost that depends on bank capital.

This paper is related to the literature on the role of public liabilities in providing liquidity by serving as stores of value. Samuelson (1958) shows that a government bond can enable intergenerational trades that would not otherwise occur. Woodford (1990) shows that when income and investment opportunities are not synchronized, investment and liquid assets are complements, so that under some conditions the issuance of government debt will "crowd in" investment. Kiyotaki and Moore (2012) consider both public and private liquidity in a similar framework, and Farhi and Tirole (2012) introduce bubbles as a store of value to explore the interplay between public, private, and bubble liquidity. In all of these models, agents would like to transfer funds forward in time, but are unable to

[6]Other papers that discuss liquid asset creation by banks in comparison to government liabilities include Stein (2012) and Greenwood, Hanson, and Stein (2010).

do so due to a market incompleteness. I build on these models by introducing financial intermediaries that supply liquidity. There have also been several empirical studies of the role of government liquidity. Krishnamurthy and Vissing-Jorgensen (2012) find that government bonds hold a liquidity premium over corporate bonds, with about half of the 100 bps average spread explained by the superior liquidity of Treasury bonds. Kashyap and Stein (2000) find evidence of a credit channel of monetary policy.

The idea that bank lending is subject to agency costs has many precedents in the banking literature. Calomiris and Kahn (1991) argue that demandable debt is a mechanism to ensure the cooperation of banks. They argue that agency problems are paramount in banking, writing "...studies of banking failures give fraud a prominent place in the list of causes. Studies of 19th- and 20th-century banking indicate that fraud and conflicts of interest characterize the vast majority of bank failures for state and nationally chartered banks." Likewise, Diamond and Rajan (2001) theorize that banks adopt fragile asset structures as a commitment device. Mattesini, Monnet, and Wright (2009) examine which agents will serve as banks in a mechanism design framework, and find that agents who have a larger stake in the system serve as banks, since exclusion from future trades is more costly for such agents. Such arguments provide a microfoundation for my assumption that agency costs are decreasing in bank capital, since bank capital serves as collateral and well-capitalized banks have more to lose if their reputations are damaged. Adrian and Shin (2010) presents evidence of procyclical leverage by financial intermediaries, and examines the implications of this for asset pricing. Gertler and Karadi (2011) presents a model in which intermediaries face financial constraints on lending tied to balance sheets, and in which the central bank can substitute for this intermediation role. They use this to justify the various special lending facilities established by the Fed during the 2008 financial crisis. This is similar to my finding that public liquidity may substitute for private liquidity, but my model would correspond to an expansion of the government's total balance sheet, e.g. through the issuance of new bonds.

This paper is also related to the recent literature exploring the connection between the 2008 financial crisis and the market for liquidity. Pozsar et al. (2013) analyzes the 2008 financial crisis as a run on the "shadow banking" system, and discusses Fed policy in response to these events. Pozsar (2013) argues that the rise of shadow banking was driven by high demand for safe and liquid secured assets similar to Treasury debt, primarily driven by institutional cash pools. In the years immediately preceding the crisis, the demand for safe liquid assets exceeded the supply of government liabilities by at least \$1.5 trillion, and the shadow banking sector developed to fill this need. Pozsar (2013) recommends that policy makers consider issuing a greater volume of Treasury bills to fill this demand for liquidity.[7] Rösch and Kaserer (2013) document the decline in market liquidity during the financial crisis.

There is a large literature exploring the linkages between the financial sector and the real economy.[8] This literature has grown greatly since the global financial crisis, and a survey is beyond the scope of this paper. Most of this literature has focused on the supply of credit to finance investment, whereas my paper focuses specifically on the role of the financial sector in providing liquid assets as means of saving.

2 Model Without Banks

I first consider the model without banks. I give the derivation in detail because it will serve as the basis for the model with banks that follows.

2.1 Preferences, Endowment, and Technology

I consider an economy containing three periods, labeled $t = 0, 1, 2$. The economy is populated by a unit measure of two types of agents: households and firms. There is a single good used for both consumption and investment, which is not storable between periods. All agents have linear utility over consumption across all three periods, i.e. they have utility functions $u(c_0, c_1, c_2) = c_0 + c_1 + c_2$.

Households have an endowment of the good in each period that is sufficiently large that their ability to lend to firms is never limited in the constrained case. I denote their period t endowment by H_t.[9] Households also enter period 0 holding a stock $\bar{\ell}$ of trees that yield a unit return of the good in period 2. These trees are the economy's stock of liquid assets, which I will refer to as outside liquidity. Firms have an initial endowment of A units of the good in period 0, and no endowment in any other period.

Firms operate a linear production technology which yields a gross return of $\rho_1 > 1$ between period 0 and period 2. Thus if a firm invests I units of the good in period 0, the project will produce $\rho_1 I$ in period 2 if the project is completed. During period 1, each firm receives an idiosyncratic shock ρ. A firm that suffers shock ρ must supply an additional ρI

[7] Bernanke et al. (2011), Caballero (2010), and Acharya and Schnabl (2010) also discuss the role of high demand for safe assets in the run-up to the 2008 crisis.

[8] Prominent early examples include Bernanke and Gertler (1989), Kiyotaki and Moore (1997), and Bernanke, Gertler, and Gilchrist (1999).

[9] At the unconstrained optimum defined below, this endowment will limit investment. The assumption here is that H_t is sufficiently large relative to firm assets that for a non-trivial leverage constraint, firm borrowing will not be limited by available funds, but only by the firm's ability to commit to repaying their investors.

units of the good to the project to continue its operation. If these funds are not provided, the project will produce nothing. ρ is drawn from $\{\rho_L, \rho_H\}$, with $\rho_L < \rho_H < \rho_1$, and takes on the value ρ_L with probability p, and ρ_H with probability $1-p$. Therefore in period 1 a measure p of firms will suffer the low shock ρ_L, and a measure $1-p$ will suffer the high shock ρ_H.

We can summarize the production plan of a firm by the initial investment I and a continuation policy rule $\lambda_s \in \{0,1\}$ for $s \in \{H, L\}$, where $\lambda_s = 1$ means that the project continues given liquidity shock s.

2.2 Unconstrained Optimum

I now characterize the unconstrained optimum. Consider a consumption plan $\{C_1, C_2, C_3\}$, where C_t is total consumption of firms and households in period t. The optimal production plan maximizes total consumption subject to the economy's resource constraints, which are

$$C_0 + I \leq A + H_0 \tag{2.1}$$

$$C_1 + p\lambda_L\rho_L I + (1-p)\lambda_H\rho_H I \leq H_1 \tag{2.2}$$

$$H_2 + p\lambda_L\rho_1 I + (1-p)\lambda_H\rho_1 I + \bar{\ell} \geq C_2 \tag{2.3}$$

In period 0, households have endowment H_0 and firms have endowment A. These funds are spent on consumption $C_0 \geq 0$ and investment $I \geq 0$. In period 1, households have endowment H_1. Funds $C_1 \geq 0$ are used for consumption, funds $p\lambda_L\rho_L I$ are used to meet low liquidity shocks, and funds $(1-p)\lambda_H\rho_H I$ are used to meet high liquidity shocks. In period 2, households have endowment H_2 and earn return $\bar{\ell}$ from their holdings of outside liquidity, and firms produce $p\lambda_L\rho_1 I + (1-p)\lambda_H\rho_1 I$. These funds are spent on consumption $C_2 \geq 0$.

The unconstrained optimal production plan is the solution to

$$\begin{aligned} \max_{\lambda, I} \; & \{C_0 + C_1 + C_2\} \\ \text{s.t.} \quad & (2.1) - (2.3) \\ & C_i \geq 0,\, I \geq 0,\, \lambda_L \in \{0,1\},\, \lambda_H \in \{0,1\} \end{aligned} \tag{2.4}$$

Proposition 1 (Optimal Production Plan)**.** *The unconstrained optimal production plan is*

$\lambda_H = 1$, $\lambda_L = 1$, and

$$I = \begin{cases} A + H_0 & R_1 \geq 0 \\ 0 & R_1 < 0 \end{cases} \tag{2.5}$$

where $R_1 = p(\rho_1 - \rho_L) + (1-p)(\rho_1 - \rho_H) - 1$.

Proof. See appendix A.[10] □

Since utility is linear with no discounting, any distribution of consumption between agents is Pareto efficient as long as total consumption is maximized. Here R_1 is the net expected return on investment. Since agents are indifferent between consuming in periods 0, 1, or 2, as long as there is a positive expected return to investment, the optimal production plan is to invest all available resources in period 0. Since both liquidity shocks are smaller than the final output of the project ρ_1, if the project has been undertaken it is profitable to meet any liquidity shock that occurs and bring the project to completion.

I assume for the rest of the paper that the project is profitable even if only the low shock triggers continuation, meaning

$$p(\rho_1 - \rho_L) > 1 \tag{2.6}$$

Condition (2.6) together with $\rho_1 > \rho_H$ implies $R_1 > 0$, so investment yields a positive return in expectation. Thus the unconstrained optimal production plan is to invest all available resources as defined in (2.5).

2.3 Limited Pledgeability

I assume that firm borrowing is subject to a limited pledgeability constraint that makes it impossible to implement the first-best production plan. This constraint arises from moral hazard. At the end of period 1, each firm with a functioning project is presented with an alternative opportunity. If a firm shirks by pursuing this opportunity, the firm's project fails and the firm earns private benefit BI, where $B > 0$. There is no legal recourse for investors to seize any portion of this private benefit. Therefore in every state in which a firm's project is successful, the firm must receive at least a share BI of the output in order to cooperate.[11]

[10] This proposition assumes that H_1 is sufficiently large relative to H_0 that the limiting factor on investment is the supply of funds in period 0, not funds available to meet the liquidity shocks in period 1. This is equivalent to the condition $H_1 \geq [p\rho_L + (1-p)\rho_H](A + H_0)$.

[11] There are many alternate motivations for limited pledgeability. This description corresponds to the microfoundation provided in Hölmstrom and Tirole (1998).

Let $\rho_0 = \rho_1 - B$ be the return on investment net the firm's outside opportunity. Since a successful project produces $\rho_1 I$ output in period 2, and since the firm must receive BI in order to operate the project, external investors may receive no more than $\rho_1 I - BI = \rho_0 I$.[12] Therefore $\rho_0 I$ is the portion of a project's final output that a firm can credibly promise to repay investors. I refer to ρ_0 as firm pledgeability, and assume it satisfies

$$\rho_0 < \rho_H \tag{2.7}$$

$$\rho_0 < 1 + p\rho_L + (1-p)\rho_H \tag{2.8}$$

$$p\rho_0 < 1 + p\rho_L \tag{2.9}$$

$$\rho_0 > p\rho_L + (1-p)\,\rho_H \tag{2.10}$$

Condition (2.7) says that pledgeable funds are insufficient to meet some liquidity shocks. Condition (2.8) and (2.9) say that expected funds required per unit of investment exceed the pledgeable portion of the return, no matter which shocks are met, so that projects cannot be financed solely with external funds. Thus limited pledgeability is sufficiently severe that it implies a "skin in the game" constraint that requires firms to put up some of their own capital. If either of these conditions failed to hold, the scale of investment would not be limited by pledgeability. Finally, condition (2.10) says that pledgeable funds are sufficient to meet the expected size of the liquidity shock, so that there are enough pledgeable funds owned by firms in the aggregate to meet all liquidity shocks in period 1. Thus liquidity will only be scarce when there is imperfect pooling of these funds. Note that (2.10) and (2.7) together imply that $\rho_0 > \rho_L$.

Households cannot borrow at all. Since they cannot offer any collateral, they will renege on any promise they make.[13]

2.4 Households

Households supply funds by purchasing state-contingent assets from firms. Given linear utility and no discounting, households are willing to purchase any asset that promises an expected return of at least 1. Therefore the supply of loanable funds in the economy is perfectly elastic at $r = 1$. Households consume all of their income that is not used to purchase assets.

[12] It is also possible to make firms' other asset holdings subject to an agency cost. Then firms would have to keep some of the return on these assets, and only a fraction could be used to repay the original investors. This is analogous to a tax on liquid assets purchased by firms, since firms would have to buy $(1+\tau)\,\ell$ assets in order to increase their pledgeable funds by ℓ. From the perspective of the firm, this is analogous to an increase in the liquidity premium.

[13] We can interpret household's holdings of outside liquidity $\bar{\ell}$ as their pledgeable funds.

Let q be the price of trees in period 0. Since trees yield a unit return, households will demand an infinite quantity of trees if $q < 1$ and no trees if $q > 1$. If $q = 1$ households are indifferent between any quantity of trees. Therefore household asset demand is perfectly elastic at $q = 1$.

2.5 Firms

Firms raise funds from households in period 0 by selling a contract offering a state-contingent return in period 2. The contract specifies payments to initial investors of $R_s^I \geq 0$ in the case of shock $s \in \{L, H\}$. These payments must be positive because households cannot commit to providing future funds. Firms also purchase ℓ trees from households at a price of q. In period 1, firms experience liquidity shocks. Firms raise funds to meet these shocks by selling new claims to households to be paid in period 2. Since households know the shock experienced by a firm, they will require exact compensation for funds provided. I denote by R_s^1 the funds repaid in period 2 to period 1 investors by a firm that experiences shock $s \in \{L, H\}$.

Any equilibrium contract must satisfy a number of constraints arising from incentive compatibility. First, the initial investors must be compensated in expectation for the funds they provide. The firm requires I funds in period 0 for its initial investment, plus $q\ell$ funds to purchase trees. Firms have initial assets A, and the rest must be raised from outside investors. In order for initial investors to buy this contract, their expected payments $pR_L^I + (1-p)R_H^I$ must satisfy

$$pR_L^I + (1-p)R_H^I \geq I + q\ell - A \tag{2.11}$$

In period 1, firms receive liquidity shocks, and those that continue production meet these shocks by issuing new liabilities to investors. I denote by R_s^1 the repayment in period 2 of a firm that experiences liquidity shock $s \in \{L, H\}$ to its period 1 investors. These funds must be sufficient to cover liquidity shocks, which implies constraints

$$\lambda_H \rho_H I \leq R_H^1 \tag{2.12}$$

$$\lambda_L \rho_L I \leq R_L^1 \tag{2.13}$$

Firms need a sufficient share of profits to cooperate. They repay outside investors from the pledgeable portion of their output and from the return on their asset holdings ℓ.

Total repayments to outside investors may not exceed

$$R_L^I + R_L^1 \leq \lambda_L \rho_0 I + \ell \tag{2.14}$$

$$R_H^I + R_H^1 \leq \lambda_H \rho_0 I + \ell \tag{2.15}$$

Note that firms do not use outside assets ℓ to pay for liquidity shocks directly. Rather, buying assets in period 0 allows firms to increase their pledgeability by the amount ℓ, and therefore borrow more from period 1 investors. These two specifications are analogous, but writing the constraints in this manner will simplify the exposition.

How can liquid assets improve the allocation? We can think of the fundamental problem as a lack of commitment on the part of households. Since projects that experience a high shock are profitable to continue ($\rho_1 > \rho_H$), it is optimal for households to provide funds to firms to meet high liquidity shocks. However, once a high liquidity shock is realized firms do not have sufficient pledgeable funds to meet them, since they can only promise a fraction $\rho_0 < \rho_H$ of their output. Ex ante, original investors would like to promise to provide funds at a loss in this event, because this would allow higher investment and therefore higher payments in good states. Since $\rho_0 > p\rho_L + (1-p)\rho_H$, the higher payments in good states can be large enough to compensate households for the negative returns in bad states so that households come out ahead on average. However, by assumption households are not able to commit to providing funds at a loss once the liquidity shock is realized.

Liquid assets provide a mechanism for circumventing this commitment problem. Rather than promising to provide funds in the future, households can provide the funds up front, and the firm can use them to purchase assets ℓ. These assets can then be used by the firm as collateral to raise additional funds in the event of a high liquidity shock. Effectively, households are providing collateral for their promise to pay firms in period 1 to meet liquidity shocks. Therefore liquid assets serve as a social commitment mechanism.

2.6 Equilibrium Contract

Firms choose the profit-maximizing contract

$$\max_{R,\lambda,I,\ell} \left\{ p\left(\lambda_L \rho_1 I - R_L^I - R_L^1 + \ell\right) + (1-p)\left(\lambda_H \rho_1 I - R_H^I - R_H^1 + \ell\right) \right\}$$

$$\text{s.t.} \quad (2.11)\text{ - }(2.15),\ R_L^I \geq 0,\ R_H^I \geq 0,\ R_L^1 \geq 0,\ R_H^1 \geq 0,\ I \geq 0$$

where $R = \{R_L^I, R_H^I, R_L^1, R_H^1\}$ and $\lambda = \{\lambda_L, \lambda_H\}$. Non-negativity constraints on payments to investors arise from the assumption of limited commitment by households.

Lemma 1. *Under the equilibrium contract, the firm always meets the low shock ($\lambda_L = 1$), and constraints (2.11) - (2.14) hold with equality.*

Proof. See appendix A. □

Intuitively, since our assumptions on parameters imply a positive return to investment, firms will invest until the pledgeability constraint is binding. Moreover, since there are sufficient funds available to finance all desired investment at a unit return, firms will exactly compensate all investors.

Lemma 1 greatly simplifies the statement of the problem. Since constraints (2.12) - (2.13) and (2.14) - (2.15) hold with equality, we can substitute them directly into the various expressions in the problem. Substituting constraints (2.14) and (2.15) into the objective function, we find that firm payoffs are $p\left(\rho_1 - \rho_0\right) I$ if the firm does not meet the high shock, and $\left(\rho_1 - \rho_0\right) I$ if the firm does. Since the pledgeability constraint binds, firms receive exactly the amount $\left(\rho_1 - \rho_0\right) I$ necessary for them to cooperate in equilibrium. Also, these equations allow us to derive exact expressions for R_s^1 and R_s^I. These are

$$\begin{aligned}
R_H^1 &= \lambda_H \rho_H I \\
R_L^1 &= \rho_L I \\
R_L^I &= \left(\rho_0 - \rho_L\right) I + \ell \\
R_H^I &= \lambda_H \left(\rho_0 - \rho_H\right) I + \ell
\end{aligned}$$

Together with (2.11), these imply leverage constraint

$$I \leq \frac{A - (q-1)\,\ell}{1 - p\left(\rho_0 - \rho_L\right) - \lambda_H (1-p)\left(\rho_0 - \rho_H\right)} \tag{2.16}$$

which will hold with equality at the optimum. Intuitively, the pledgeability constraint is sufficiently severe that firms must supply a fraction of the capital for investment (a skin-in-the-game constraint). This implies that investment is limited by firms' initial assets, which can be expressed as a limit on leverage, as given in (2.16). When $q > 1$, firms must put up some of their own funds to purchase additional liquidity, reducing available funds and thus equilibrium investment.

The non-negativity constraints on R_H^1, R_L^1, and R_L^I are trivially satisfied. The non-negativity constraint on R_H^I can be expressed as $\ell \geq \lambda_H \left(\rho_H - \rho_0\right) I$, which implies that holdings of outside liquidity ℓ must be sufficient to finance the high liquidity shock.

We can now express the optimal contracting problem of the firm as

$$\max_{\lambda_H, I, \ell} \{(\rho_1 - \rho_0)\,[p + (1-p)\lambda_H]\, I\}$$
$$\text{s.t.} \quad (2.16),\ \lambda_H\,(\rho_H - \rho_0)\, I \leq \ell$$

plus non-negativity constraints on I and ℓ. The following proposition characterizes the solution:

Proposition 2. *The optimal production plan of firms is to meet the high shock ($\lambda_H = 1$) if and only if*

$$q - 1 \leq \frac{(1-p)\,[1 - p\,(\rho_H - \rho_L)]}{p\,(\rho_H - \rho_0)} \tag{2.17}$$

If (2.17) holds with equality, the firm is indifferent between $\lambda_H = 1$ and $\lambda_H = 0$. The optimal choice of I is

$$I = \frac{A}{1 - p\,(\rho_0 - \rho_L) - \lambda_H\,(q-p)\,(\rho_0 - \rho_H)}$$

and the optimal choice of ℓ is $\ell = \lambda_H\,(\rho_H - \rho_0)\, I$.

Proof. See appendix A. □

Since $q - 1$ is always non-negative, (2.17) will never hold if the parameters satisfy $p\,(\rho_H - \rho_L) > 1$. This condition is a simplification of

$$\frac{pA}{1 - p\,(p_0 - \rho_L)} > \frac{A}{1 - p\,(\rho_0 - \rho_L) - (1-p)\,(\rho_0 - \rho_H)}$$

where the left-hand side is output when firms meet only the low shock, and the right-hand side is output when firms meet both shocks. When $p\,(\rho_H - \rho_L) \leq 1$, (2.17) defines a cutoff level of q above which it is optimal to meet only the low shock. Intuitively, firms allocate limited pledgeable funds between financing the initial investment and meeting liquidity shocks. If liquidity is sufficiently expensive, firms substitute away from the more expensive input to the production process by increasing the scale of the initial investment, and reducing the fraction of liquidity shocks they meet.

2.7 Equilibrium

Proposition 2 defines a demand for liquid assets ℓ, which is a decreasing function of q and drops to zero at the cutoff defined by (2.17). To determine equilibrium q, we impose clearing in the liquidity market. Since households will only hold assets that yield a unit return or better, if $q > 1$ aggregate liquidity demand is exactly equal to demand from firms. Equilibrium is depicted in Figure 1 and characterized in Proposition 3.

Proposition 3. *Let* $I_0 = A/\chi_0$ *and* $I_1(q-1) = A/(\chi_1 + (q-1)(\rho_H - \rho_0))$*, where* $\chi_0 = 1 - p(\rho_0 - \rho_L)$ *and* $\chi_1 = 1 - \rho_0 + p\rho_L + (1-p)\rho_H$*. Then,*

(i) If $p(\rho_H - \rho_L) > 1$*, equilibrium* $q - 1 = 0$ *and all firms choose* $\lambda_H = 0$*,* $\ell = 0$*, and* $I = I_0$*.*

(ii) If $p(\rho_H - \rho_L) \leq 1$ *and* $\bar{\ell} \geq (\rho_H - \rho_0) I_1(0)$*, equilibrium* $q - 1 = 0$ *and firms choose* $I = I_1(0)$*,* $\ell = (\rho_H - \rho_0) I_1(0)$*, and* $\lambda_H = 1$*.*

(iii) If $p(\rho_H - \rho_L) \leq 1$ *and* $p(\rho_H - \rho_0) I_0 \leq \bar{\ell} \leq (\rho_H - \rho_0) I_1(0)$*, equilibrium* $q - 1$ *is*

$$q - 1 = \frac{A}{\bar{\ell}} - \frac{\chi_1}{\rho_H - \rho_0}$$

and firms choose $\lambda_H = 1$*,* $\ell = \bar{\ell}$*, and* $I = I_1(q-1) = \bar{\ell}/(\rho_H - \rho_0)$*.*

(iv) If $p(\rho_H - \rho_L) \leq 1$ *and* $\bar{\ell} < p(\rho_H - \rho_0) I_0$*, equilibrium* q *is*

$$q - 1 = \frac{\chi_0 - p\chi_1}{p(\rho_H - \rho_0)}$$

and firms are indifferent between $\lambda_H = 0$ *and* $\lambda_H = 1$*. A fraction* ζ *of firms choose* $\lambda_H = 1$*,* $I = I_1(q-1) = pI_0$*, and* $\ell = (\rho_H - \rho_0) pI_0$*, where* $\zeta = \bar{\ell}/[(\rho_H - \rho_0) pI_0]$*. The remaining fraction* $1 - \zeta$ *choose* $\lambda_H = 0$*,* $\ell = 0$*, and* $I = I_0$*.*

Proof. See appendix A. □

Assuming $p(\rho_H - \rho_L) \leq 1$ so that it is potentially optimal to meet the high shock, Proposition 3 defines two cutoff levels of $\bar{\ell}$.[14] If outside liquidity $\bar{\ell}$ is above the higher cutoff, then all firms meet both liquidity shocks and there is no liquidity premium ($q - 1 = 0$). In this case, the economy still fails to achieve the unconstrained optimum due to the usual effect of credit constraints restricting initial investment, but there are no additional effects due to limited liquidity.

I_0 is the optimal level of investment when $\lambda_H = 0$, and $I_1(q-1)$ is the optimal level of investment when $\lambda_H = 1$, which is strictly decreasing in $q - 1$. I refer to the equilibrium allocation with $q = 1$ as the *constrained optimum*. The constrained optimum achieves the highest level of investment that respects the aggregate limited pledgeability constraint. When both shocks are met, this level of investment is $I_1(0) = A/\chi_1$, meaning that firms

[14] Note that $p(\rho_H - \rho_0) I_0 \leq (\rho_H - \rho_0) I_1(0)$ if and only if $p(\rho_H - \rho_L) \leq 1$, and so the cutoffs satisfy the assumed ordering.

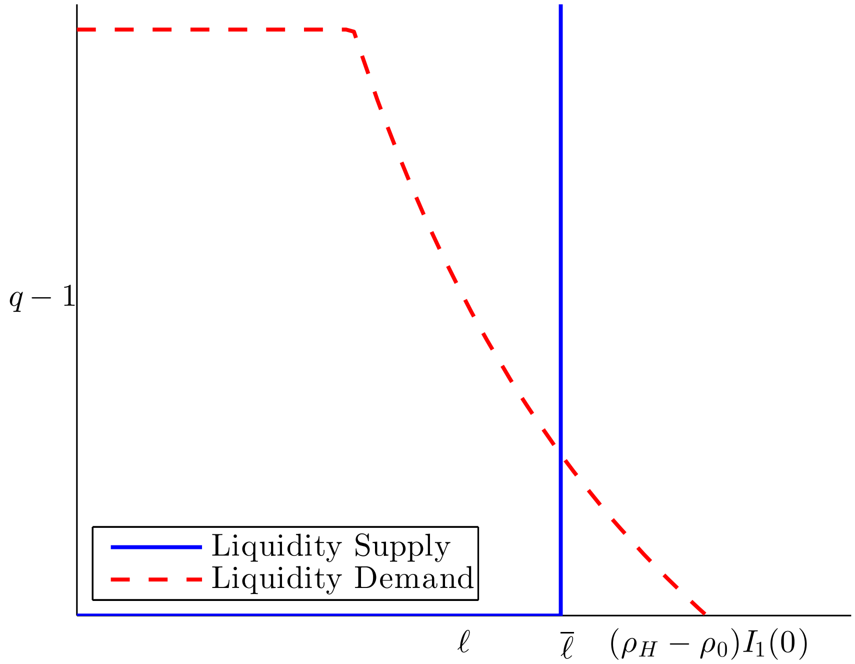

Figure 1: Equilibrium without banks.

must provide χ_1 of their own funds for every unit of investment, and so $1/\chi_1$ is the leverage ratio in the economy. $\chi_1 = 1 + p\rho_L + (1-p)\rho_H - \rho_0$ is the amount by which expected required funds per unit of investment exceed pledgeable funds. Our assumptions on ρ_0 imply $\chi_1 \in (0,1)$, and so $I \in (A, \infty)$. When ρ_0 is large relative to $1 + p\rho_L + (1-p)\rho_H$, most investment is financed using external funds and leverage is high. When ρ_0 is small relative to $1 + p\rho_L + (1-p)\rho_H$, external financing is limited and leverage is low.

If $\bar{\ell}$ lies below the higher cutoff, then the economy is liquidity constrained, in the sense that there is insufficient liquidity for all firms to meet the high shock at the constrained optimal level of investment. Therefore equilibrium q must rise until the quantity of liquidity demanded by firms equals available liquidity $\bar{\ell}$. Liquidity demand is downward sloping because firms' pledgeable funds are used to pay the higher price of liquidity, and so funds available to finance initial investment fall, tightening the leverage constraint.

If $\bar{\ell}$ lies below the lower cutoff, then there is insufficient liquidity available for all firms to meet the high shock. Intuitively, firms are indifferent between meeting and not meeting the high shock when $I_1 = pI_0$. Thus the lower cutoff is the level of $\bar{\ell}$ that is just sufficient for all firms to meet the high liquidity shock given investment $I_1 = pI_0$. The corresponding liquidity premium $q-1$ is implicitly defined by $I(q-1) = pI_0$. For $\bar{\ell}$ below this level, some firms will meet the high shock and some will not, with the fraction determined by available $\bar{\ell}$.

I refer to an equilibrium that lies strictly between these two cutoffs as an interior equilibrium. This is the most interesting case, because here a marginal change in available liquidity will change both the equilibrium investment and the equilibrium liquidity premium q. Such a shift is depicted in Figure 2, which shows that a fall in outside liquidity $\bar{\ell}$ will increase the equilibrium liquidity premium $q - 1$ and decrease equilibrium investment I.

In the remainder of the paper, I assume that $p(\rho_H - \rho_L) < 1$, so that firms strictly prefer to meet the high shock if there is no liquidity premium.

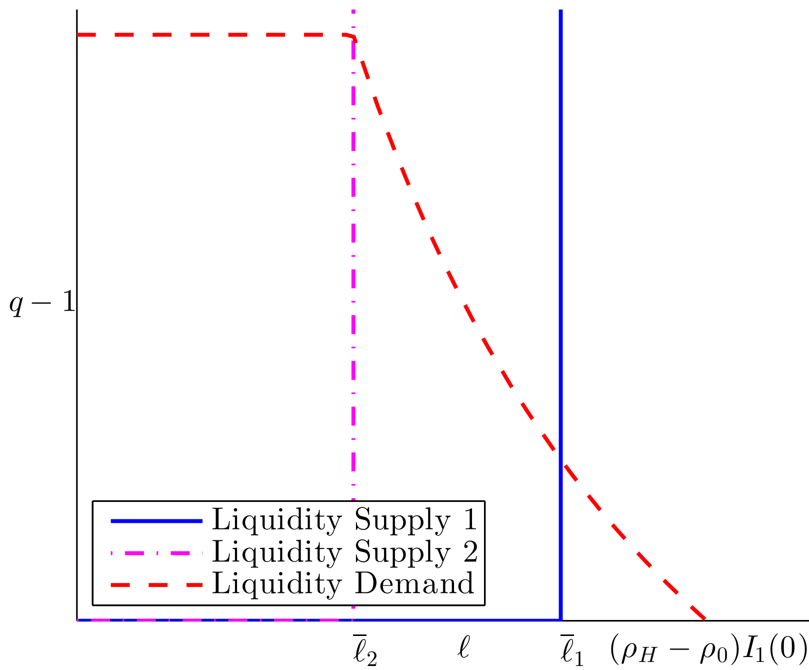

Figure 2: Fall in outside liquidity without banks.

3 Model With Banks

In the previous section, I showed that there may be insufficient liquidity in the economy, resulting in lower investment and, in some cases, a lower share of firms meeting high liquidity shocks. But by assumption, there are sufficient pledgeable funds in the aggregate to meet all liquidity shocks, since firms that experience a low shock have more pledgeable funds available than they need, whereas firms that experience a high shock are short pledgeable funds. Thus firms can achieve the constrained optimum if they are able to pool their pledgeable funds by arranging for transfers from firms with excess funds to

Period 0	Period 1	Period 2
• Firms buy credit lines from banks with maximum M. • Banks receive firm shares worth $(1-p)M+\pi$.	• Measure $1-p$ of firms receive high shock, receive M each from banks. • Banks raise $(1-p)M$ from households, pay to firms. • Households receive bank claims worth $(1-p)M$.	• Measure $1-p$ of firms pay $R_H^B = 0$ each to banks. • Measure p of firms pay $R_L^B = ((1-p)M+\pi)/p$ each to banks. • Banks pay $(1-p)M$ to households, keep π.

Figure 3: Timeline of bank credit lines.

firms that receive a high shock and need additional funds.

To implement this liquidity pooling, I introduce financial intermediaries (banks) into the model.[15] Banks sell financial contracts in period 0 to firms in exchange for shares in the firm. These contracts obligate banks to provide M funds in period 1 to firms that experience a high liquidity shock. Banks obtain these period 1 funds from households by borrowing against the future returns on their portfolio. Since firms that experience a high liquidity shock will dilute their period 0 shares with new borrowing, banks take losses on these credit lines. To make up for this, banks require higher payments from firms that experience the low shock. The resulting pattern of payments between banks, firms, and households is shown in Figure 3.

I call these arrangements *credit lines*, since they are analogous to the credit lines in Hölmstrom and Tirole (1998). Their key feature is that they pool pledgeable funds via effective transfers from firms that experience low liquidity shocks to firms that experience high shocks, intermediated by banks. The most straightforward implementation of this arrangement is a system of direct contingent transfers, as described above. This arrangement can be interpreted as an insurance contract.

Alternatively, the same set of net payments can be implemented using demand deposits. In this case, firms raise additional initial funds and deposit them with banks, and banks invest these funds by purchasing shares in firms. In period 1, firms withdraw funds from their accounts to meet liquidity shocks, and banks raise new funds from households to finance these withdrawals. Another implementation is a contract that resembles a corporate credit line. Under this arrangement, firms pay an upfront fee in exchange for

[15] I assume that firms cannot implement liquidity pooling by themselves.

access to a line of credit at a specified rate, together with a fee on unused credit.[16]

If banks are not subject to any commitment frictions, they will be able to perfectly insure against idiosyncratic liquidity shocks, and the economy will achieve the constrained optimal level of investment, even when there is no outside liquidity ($\bar{\ell} = 0$). However, it is reasonable to think that banks are subject to agency costs of a similar nature to firms. Banks act as agents that invest on behalf of their depositors, in this case households that provide them with funds in period 1. In the process of lending, banks may need to exert effort in screening firms, verifying the liquidity shock, and collecting funds. Banks also have opportunities to defraud their depositors by withholding funds. If banks do not receive sufficient profits relative to the size of their portfolios, they will not exert full effort screening or collecting loans, and may have an incentive for fraud.

Let bank agency costs be represented by the function $C(\varphi M, K)$, where M is the payment in period 1 to firms that experience the high liquidity shock, φ is the measure of credit lines sold by the bank, and K is bank equity at the start of period 2. I assume that K is exogenous and known to all agents at the beginning of period 0. In order to commit to properly intermediating a measure φ of credit lines of size M, a bank with equity K must receive at least $C(\varphi M, K)$ in profits, i.e.

$$\varphi\pi \geq C(\varphi M, K)$$

where π is profits received per credit line sold. This is the bank's incentive compatibility constraint.

Letting $D = \varphi M$ denote total funds intermediated, I assume that the agency cost function $C(\cdot)$ is twice continuously differentiable at all points (D, K), with $D \geq 0$ and $K > 0$. I assume that agency costs are increasing and convex in funds intermediated, $C_1(\cdot) > 0$ and $C_{11}(\cdot) > 0$, and decreasing in bank equity, $C_2(\cdot) < 0$. Finally, I assume that $\lim_{D\to 0} C_1(D, K) = 0$, $\lim_{D\to\infty} C_1(D, K) = \infty$, $C(0, K) = 0$ for $K \geq 0$, and $C(D, 0) = \infty$ for $D > 0$.

3.1 Preferences, Endowment, and Technology

Preferences, endowments, and technology are the same as in section 2, except for the inclusion of banks. Like firms and households, there is a unit measure of banks, and banks have linear utility over consumption across all three periods, i.e. $u(c_0, c_1, c_2) = c_0 + c_1 + c_2$. As described above, banks have equity K entering period 2.

[16]This is a common arrangement for corporate credit lines, as described by Sufi (2009).

3.2 Households

As before, households supply funds perfectly elastically at an expected return of 1. Therefore every asset will be priced at its expected return, or households will not hold that asset. Households are willing to sell their supply of trees at a price $q \geq 1$, and will not buy any trees if $q > 1$.

3.3 Firms

In period 0, firms choose production plan $\{I, \lambda\}$, with $I \geq 0$ investment in period 0, and project continuation policy rule λ_s for state $s \in \{L, H\}$. Firms raise funds in period 0 by selling shares offering a state-contingent return in period 2. The contract specifies payments to initial investors of R_s^I in the case of shock $s \in \{L, H\}$. Firms also purchase ℓ trees from households at a price of q.

Altogether, the firm requires total funds $I + q\ell$ in period 0. It has its own funds A, and raises the remainder from outside investors. Since there is a unit cost to external funds, expected payments to initial investors must satisfy

$$pR_L^I + (1-p)R_H^I \geq I + q\ell - A \tag{3.1}$$

In period 0, firms also choose whether to purchase credit lines from banks. I represent this decision by $\lambda_B \in \{0, 1\}$. The equilibrium credit line consists of a pair (π, M), where M is the maximum amount that can be drawn on in period 1, and π is the expected profits received by the bank. The firm pays for the credit line with shares that yield net payments of R_s^B in period 2 in the event of shock s. These payoffs must satisfy

$$pR_L^B + (1-p)R_H^B \geq \pi + (1-p)M \tag{3.2}$$

In period 1, firms experience liquidity shocks. Firms raise funds to meet these shocks by issuing new shares offering a return in period 2. I denote by R_s^1 the funds repaid in period 2 to period 1 investors by a firm that experiences shock s. Firms that experience a low shock raise funds ρ_L, whereas firms that experience a high shock raise new funds $\rho_H - \lambda_B M$ and draw on their credit lines for the remainder. Since there is a unit cost of funds in period 1, these repayments must satisfy

$$R_L^1 \geq \lambda_L \rho_L I \tag{3.3}$$

$$R_H^1 \geq \lambda_H \rho_H I - \lambda_B M \tag{3.4}$$

Firms repay their outside investors from the pledgeable portion of their income and from the return on their asset holdings ℓ. Thus total repayments in each state cannot exceed available pledgeable funds

$$R_L^I + R_L^1 + \lambda_B R_L^B \leq \lambda_L \rho_0 I + \ell \tag{3.5}$$

$$R_H^I + R_H^1 + \lambda_B R_H^B \leq \lambda_H \rho_0 I + \ell \tag{3.6}$$

Firms choose production to solve

$$\max_{R,\lambda,I,\ell} \left\{ p \left(\lambda_L \rho_1 I - R_L^I - R_L^1 - \lambda_B R_L^B + \ell \right) + (1-p) \left(\lambda_H \rho_1 I - R_H^I - R_H^1 - \lambda_B R_H^B + \ell \right) \right\}$$
$$\text{s.t.} \quad (3.1) - (3.6),\ \{R, I\} \geq 0$$

where $R = \left\{ R_L^I, R_H^I, R_L^1, R_H^1, R_L^B, R_H^B \right\}$. Non-negativity constraints on payments to households arise from the assumption of limited commitment by households.

Lemma 2. *Under optimal firm behavior, $\lambda_L = 1$ and constraints (3.1) - (3.6) hold with equality.*

Proof. See appendix B. □

The intuition for this result is similar to Lemma 1. Since investment yields a positive return, firms will borrow until the pledgeability constraints are binding. The other constraints bind because this minimizes payments to creditors.

Lemma 2 greatly simplifies the statement of the problem. Since (3.5) - (3.6) hold with equality, we can substitute them into the objective function to find that firm payoffs are $p(\rho_1 - \rho_0) I$ if $\lambda_H = 0$, and $(\rho_1 - \rho_0) I$ if $\lambda_H = 1$. Since the pledgeability constraint binds, firms receive exactly the amount $(\rho_1 - \rho_0) I$ necessary for them to cooperate in equilibrium.

Taking a weighted sum of (3.5) and (3.6), we have:

$$p \left(R_L^I + R_L^1 + \lambda_B R_L^B \right) + (1-p) \left(R_H^I + R_H^1 + \lambda_B R_H^B \right) = p(\rho_0 I + \ell) + (1-p)(\lambda_H \rho_0 I + \ell)$$

Substituting in (3.1) - (3.4), we obtain the leverage constraint

$$I = \frac{A - (q-1)\ell - \lambda_B \pi}{1 - p(\rho_0 - \rho_L) - \lambda_H (1-p)(\rho_0 - \rho_H)} \tag{3.7}$$

Intuitively, the pledgeability constraint is sufficiently severe that firms must supply a fraction of the capital for investment (a skin-in-the-game constraint). This implies that investment is limited by firms' initial assets, which can be expressed as a constraint on leverage.

We can now rewrite the other constraints in the problem. The non-negativity constraints on R^1_L, R^1_H, R^I_L and R^B_L are trivially satisfied, given $M \geq 0$. The non-negativity constraints on R^I_H and R^B_H imply

$$\lambda_H(\rho_H - \rho_0)I \leq \lambda_B M + \ell \tag{3.8}$$

Thus the firm problem reduces to

$$\max_{\lambda_H,\lambda_B,I,\ell} \{(\rho_1 - \rho_0)\,[p + (1-p)\lambda_H]\,I\}$$
$$\text{s.t.} \quad (3.7) \text{ and } (3.8)$$

Optimal firm behavior is characterized by Proposition 4.

Proposition 4. *Let $q \geq 1$ and $(M, \pi) \geq 0$ with $M\chi_1 \leq (\rho_H - \rho_0)(A - \pi)$ be given.*[17] *Let $\chi_1 = 1 - \rho_0 + p\rho_L + (1-p)\,\rho_H$ and $\chi_0 = 1 - p(\rho_0 - \rho_L)$. Then,*

(i) Firms will choose $\lambda_B = 1$ if and only if

$$\frac{\pi}{M} \leq q - 1 \tag{3.9}$$

and will be indifferent between $\lambda_B = 0$ and $\lambda_B = 1$ if (3.9) holds with equality.

(ii) Firms will choose $\lambda_H = 1$ if and only if

$$\frac{A - \lambda_B(\pi - (q-1)M)}{\chi_1 + (q-1)(\rho_H - \rho_0)} \geq \frac{pA}{\chi_0} \tag{3.10}$$

and will be indifferent between $\lambda_H = 0$ and $\lambda_H = 1$ if (3.10) holds with equality.

(iii) Firms will invest

$$I = \frac{A - \lambda_H\lambda_B\,[\pi - (q-1)M]}{1 - p(\rho_0 - \rho_L) - \lambda_H(q-p)(\rho_0 - \rho_H)}$$

and will purchase outside liquidity

$$\ell = \lambda_H\left((\rho_H - \rho_0)I - \lambda_B M\right)$$

Proof. See appendix B. □

[17] The condition $M\chi_1 \leq (\rho_H - \rho_0)(A - \pi)$ rules out cases where M is larger than the amount of liquidity any firm seeks to hold.

Since by Lemma 2 all constraints bind and firms always meet the low liquidity shock, the only decisions that remain are whether to meet the high liquidity shock, and whether to purchase a credit line from banks. Proposition 4 says that firms hold credit lines as long as they are cheaper than outside liquidity. Since the unit cost of outside liquidity is q, whereas the unit cost of credit lines is $1 + \pi/M$, this implies condition (3.9).

Optimal investment when firms do not meet the high shock is $I_0 = A/\chi_0$, and when firms meet the high shock is

$$I_1(q-1) = \frac{A - \lambda_B[\pi - (q-1)M]}{\chi_1 + (q-1)(\rho_H - \rho_0)}$$

Firm profits are proportional to $(p + \lambda_H(1-p))I$, and so firms will meet the high shock as long as $I_1 > pI_0$, which is condition (3.10). Since $\rho_H > \rho_0$, meeting the high shock means there are less pledgeable funds available per unit invested, and so the leverage constraint (3.7) implies $I_1 < I_0$. Moreover, a higher price of liquidity will raise the costs of meeting the high shock, further reducing I_1.

3.4 Banks

There is a unit measure of banks that sell credit lines to firms. I assume that bank issuance of credit lines is perfectly diversified across firms. Thus although there is a unit measure of each agent, in a sense there are "more" firms than banks.[18]

Banks sell credit lines to firms, which are contracts represented by $\left(M, R_L^B, R_H^B\right)$. M is the credit maximum, which are the funds the firm may draw on in period 1 to meet a high liquidity shock. R_L^B and R_H^B are net payments from the firm to the bank in period 2 in case of a low shock and high shock respectively.

If all firms purchase credit lines and both shocks are met, then in period 1 a total of $(1-p)M$ funds will be supplied by banks to firms that received a high shock. Banks obtain these funds by selling claims to households, which they repay in period 2. Each bank receives payments $pR_L^B + (1-p)\,R_H^B$ from firms in period 2, and it must compensate its depositors by repaying the $(1-p)\,M$ funds raised in period 1. For notational convenience, I suppose that firms that did not draw on their credit lines in period 1 receive the funds M from the bank in period 2. Since R_H^B was defined as the net payments to banks, bank expected profits for each credit line are

$$\pi = p(R_L^B - M) + (1-p)\,R_H^B - M$$

[18] We can formalize this notion by supposing there are N banks of measure $1/N$, and N^2 firms of measure $1/N^2$, so that each bank has N firms as customers, and letting $N \to \infty$.

As described in the beginning of this section, banks must be sufficiently compensated to cooperate. For each credit line issued, banks repay funds M and receive profits π, which must satisfy

$$\varphi\pi \geq C(\varphi M, K) \tag{3.11}$$

where φ is the measure of firms to which a particular bank issues credit lines.

3.5 The Equilibrium Credit Line

What will be the equilibrium credit line? Suppose that rather than selling credit lines of a particular price and size (π, M), banks set a unit price ϑ, and sell any credit line with $(\vartheta M, M)$. Thus $\vartheta = \pi/M$ is equal to profits received by banks per unit of liquidity provided. Let $D = \varphi M$ be total funds intermediated by the bank, and let $\Pi = \varphi\pi = \vartheta D$ be profits from selling this quantity of credit lines.

From Proposition 4 we know that firms will never purchase a credit line at price $\pi/M = \vartheta > q - 1$. If credit lines are offered at different prices, firms will naturally prefer to purchase the one with the lowest price ϑ. However, firms will not purchase a credit line from a bank whose agency constraint is violated, since such a bank would not honor its obligations. This implies that a bank's total sale of credit lines D must satisfy $\vartheta D \geq C(D, K)$.

Let $\tilde{D}(\vartheta)$ denote the largest value of D for which $\vartheta D \geq C(D, K)$. Given our assumptions $C_{11} > 0$, $C(0, K) = 0$, and $C(D, K) \to \infty$ as $t \to \infty$, $\vartheta\tilde{D}(\vartheta) = C(\tilde{D}(\vartheta), K)$ will hold with equality, and $\tilde{D}(\cdot)$ satisfies $\tilde{D}(0) = 0$ and $\tilde{D}'(\vartheta) > 0$. Then we may write the constraint $\vartheta D \geq C(D, K)$ as $D \leq \tilde{D}(\vartheta)$.

Even if banks set a price for credit lines $\vartheta < q - 1$, firms may not purchase all the way up to $\tilde{D}(\vartheta)$. This is because there is a maximum amount of liquidity firms desire to hold. When the price of liquidity is such that all firms meet both liquidity shocks, maximum desired liquidity satisfies

$$\bar{D}(\vartheta) = \frac{A(\rho_H - \rho_0)}{\chi_1 + \vartheta(\rho_H - \rho_0)}$$

Now we can define the demand for credit lines facing a particular bank i, which we denote by $D^d(\vartheta_i)$. Suppose that the price of outside credit is q and the price of credit lines set by other banks is ϑ. Then demand for credit lines satisfies

Definition 1 (Demand for Credit Lines)**.** Demand for credit lines $D^d(\vartheta_i)$ satisfies

(i) $D^d(\vartheta_i) = 0$ if $\vartheta > q - 1$, or $\vartheta_i > \vartheta$ and $\tilde{D}(\vartheta) \geq \bar{D}(\vartheta)$.

(ii) $D^d(\vartheta_i) = \tilde{D}(\vartheta_i)$ if $\vartheta \leq q-1$ and either $\vartheta_i < \vartheta$, or $\vartheta_i \geq \vartheta$ and $\tilde{D}(\vartheta) < \bar{D}(\vartheta)$.

(iii) $D^d(\vartheta_i) = \bar{D}(\vartheta_i)$ if $\vartheta_i \leq q-1$ and $\vartheta_i = \vartheta$ and $\tilde{D}(\vartheta) = \bar{D}(\vartheta)$.

Clearly no firm will purchase a credit line with a cost above $q-1$. If $\vartheta_i \leq q-1$ we may distinguish three cases. First if $\vartheta_i < \vartheta$, then the bank underprices its competition and sells up to its agency costs $D^d(\vartheta_i) = \tilde{D}(\vartheta^i)$. Second, if $\vartheta_i = \vartheta$, then a bank will sell up to its agency costs if there is enough total demand to go around, but if not it will receive an equal share of firm's desired credit lines, i.e. $D^d(\vartheta_i) = \min\left(\tilde{D}(\vartheta_i), \bar{D}(\vartheta)\right)$. Finally, if a bank chooses $\vartheta_i > \vartheta$, then it will receive any residual demand after firms have purchased from all other banks. Thus if $\bar{D}(\vartheta) > \tilde{D}(\vartheta)$, then $D^d(\vartheta_i) = \tilde{D}(\vartheta)$, and if not then $D^d(\vartheta_i) = 0$.

We can write the bank pricing problem as

$$\max_{\vartheta_i} \vartheta_i D^d(\vartheta_i)$$

Proposition 5 describes the equilibrium credit line.

Proposition 5 (Equilibrium Credit Line)**.** *Given price of outside liquidity q, the equilibrium credit line satisfies*

(i) If $\tilde{D}(q-1) \leq \bar{D}(q-1)$, then $\vartheta = q-1$ and firms purchase $D = \tilde{D}(q-1)$ credit lines.

(ii) If $\tilde{D}(q-1) > \bar{D}(q-1)$, then ϑ satisfies $\tilde{D}(\vartheta) = \bar{D}(\vartheta)$, and firms purchase $D = \bar{D}(\vartheta)$ credit lines.

Proof. See appendix B. □

3.6 Equilibrium

We are now able to define equilibrium. Given price of outside liquidity q, Proposition 5 defines the optimal credit line (π, M), and Proposition 4 defines optimal firm behavior, including holdings of outside liquidity $\ell(q)$.

To define equilibrium, we need one further condition to determine q, for which we use the outside liquidity market clearing condition. Let ζ be the fraction of firms that meet the high shock, and suppose that each of these firms purchase credit lines (π, M) and hold liquidity ℓ. Since they have no need of liquidity, households only desire to hold outside liquidity when $q = 1$. Therefore either $q = 1$ and firms may hold any ℓ such that $\zeta\ell \leq \bar{\ell}$, or else $q > 1$ and firms must hold $\zeta\ell = \bar{\ell}$. We can express this market clearing condition as

$$(q-1)\left(\bar{\ell} - \zeta\ell\right) = 0 \tag{3.12}$$

This equilibrium is described in Proposition 6.

Proposition 6. *Let $K > 0$ and $\bar{\ell} \geq 0$ be given, and let $\chi_1 = 1 - \rho_0 + p\rho_L + (1-p)\rho_H$ and $\chi_0 = 1 - p(\rho_0 - \rho_L)$. Let $I_0 = A/\chi_0$ and $I_1(q-1) = A/(\chi_1 + (q-1)(\rho_H - \rho_0))$. Let M_1 be defined implicitly by $(\rho_H - \rho_0)I_1(\frac{C(M_1,K)}{M_1}) = M_1 + \bar{\ell}$, and let M_2 be defined by $I_1(\frac{C(M_2,K)}{M_2}) = pI_0$. Then,*

(i) If $\bar{\ell} \geq (\rho_H - \rho_0)I_1(0)$, then $\zeta = 1$, $I = I_1(0)$, $q = 1$, $M = \pi = 0$, and $\ell = (\rho_H - \rho_0)I_1(0)$.

(ii) If $\bar{\ell} < (\rho_H - \rho_0)I_1(0)$, then M_1 and M_2 are uniquely defined, and if $M_1 \leq M_2$, equilibrium satisfies $M = M_1$, $q - 1 = \frac{C(M,K)}{M}$, $\ell = \bar{\ell}$, $\zeta = 1$, $\pi = (q-1)M$, and $I = I_1(0)$.

(iii) If $\bar{\ell} < (\rho_H - \rho_0)I_1(0)$ and $M_1 > M_2$, then equilibrium satisfies $q - 1 = \frac{C(M_2,K)}{M_2}$, $I = I_1(q-1) = pI_0$, $\zeta = \frac{M_2 + \bar{\ell}}{(\rho_H - \rho_0)I}$, $\ell = \bar{\ell}/\zeta$, $M = M_2/\zeta$, and $\pi = (q-1)M$.

Proof. See appendix B. □

Once again, I_0 is optimal investment when $\lambda_H = 0$, and $I_1(q-1)$ is optimal investment when $\lambda_H = 1$, which is decreasing in $q - 1$.

If there is sufficient outside liquidity to meet all liquidity needs at the constrained optimal level of investment $I_1(0) = A/\chi_1$, then there is no need for banks to supply credit lines and there is no liquidity premium.

If there is insufficient outside liquidity to finance all shocks, then banks sell credit lines. Banks will require some profits to cooperate, and arbitrage between bank lines of credit and outside liquidity will set $q - 1$ equal to average bank agency costs. Thus there is a positive liquidity premium. Since households do not value outside liquidity for its liquidity properties, they are unwilling to hold it if there is a positive liquidity premium, and so all outside liquidity will be held by firms.

When there is a positive liquidity premium, it may be the case that all firms meet the high liqudity shock, or that only a fraction of firms meet it. To distinguish these cases, we compute two levels of M. M_1 corresponds to the equilibrium credit line that would prevail if all firms met the high liquidity shock. Firms will choose to meet the high liquidity shock as long as investment $I_1(q-1) \geq pI_0$. M_2 corresponds to the highest level of M for which this expression holds when all firms meet the high shock.

We can illustrate equilibrium by means of a supply and demand diagram in the market for liquidity, as depicted in Figure 4. Total liquidity is $M + \bar{\ell}$. Since households are willing to sell their liquidity holdings at any price $q \geq 1$, the supply of liquidity is horizontal at $q = 1$ up to $\bar{\ell}$. Liquidity above this level is provided by banks, who issue credit lines that provide total liquidity $D(q)$, which is implicitly defined by $q - 1 = C(D,K)/D$.

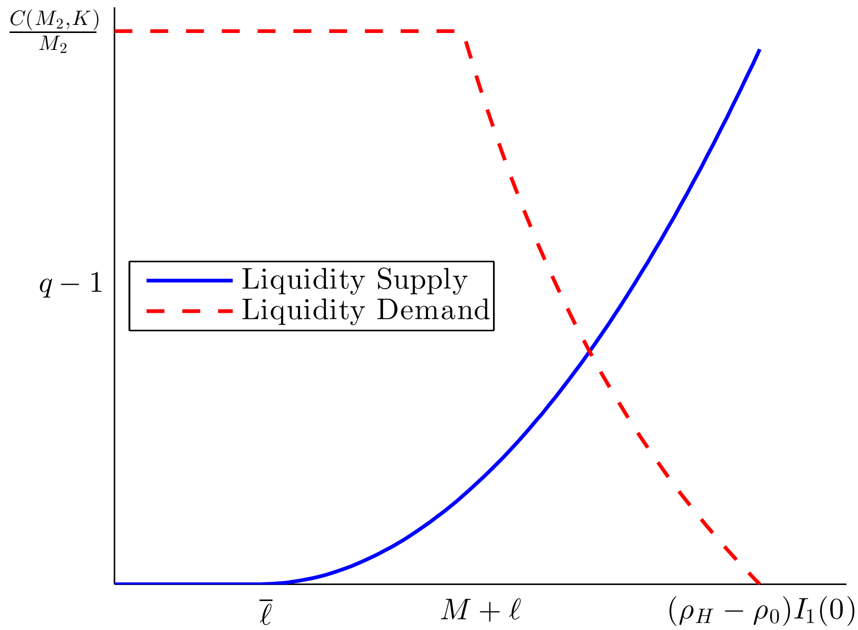

Figure 4: Equilibrium with banks.

Demand for liquidity equals $\zeta(q)(\rho_H - \rho_0)I_1(q)$. This is downward sloping with x-intercept at $(\rho_H - \rho_0)I_1(0)$ up to the value of q at which $I_1(q) = pI_0$. At this q firms are indifferent with respect to meeting the high liquidity shock, and so demand for liquidity is horizontal.

Proposition 6 shows that if there is insufficient outside liquidity $\bar{\ell}$, there will be a positive liquidity premium and lower investment in equilibrium compared to the constrained optimum. Thus in the presence of bank agency costs, the addition of banks to the model does not make outside liquidity unnecessary. This is worth emphasizing because it is the point of departure from Hölmstrom and Tirole (1998). Hölmstrom and Tirole (1998) do not have bank agency costs, and thus the introduction of banks allows the economy to achieve the constrained optimum. In order to allow for a non-trivial discussion of liquidity and optimal policy in the presence of banks, Hölmstrom and Tirole (1998) introduce an aggregate liquidity shock. By including bank agency costs, I find that liquidity can be scarce in the presence of banks even when there is no aggregate liquidity shock.

Moreover, equilibrium investment is determined by the liquidity premium, which in equilibrium equals the average agency costs of banks. Since firms can meet liquidity shocks either using outside liquidity or credit lines, these assets must have the same price by arbitrage. Intuitively, we can think of banks as a sector of the economy that produces liquid assets, and agency costs define the production function of this sector. The precise nature of agency costs in the banking sector determines the liquidity premium, which is a component of the overall cost of the asset, along with its riskiness and return. Thus anything that affects agency costs in the banking sector will affect the capacity of the economy to commit funds, which will in turn affect investment and output.

Note that when $K = 0$, the equilibrium is as in section 2 without banks. Thus the specification with banks nests the specification without banks. In this case the supply of liquidity would be vertical at $\bar{\ell}$, so that $M(q) = 0$ for all q.

4 Comparative Statics

I now explore the properties of the equilibrium defined in Proposition 6. Given our discussion of the role of liquidity, we are interested in how changes in the supply of liquidity affect equilibrium values, notably investment I and the liquidity premium $q - 1$. I explore these questions by deriving and discussing the comparative statics of the equilibrium with respect to changes in outside liquidity $\bar{\ell}$ and bank capital K, which are the two determinants of private liquidity.

As it is the most interesting case, we restrict the following discussion to the case with $\bar{\ell} < \frac{(\rho_H - \rho_0)A}{\chi_1}$ and $M_1 > M_2$, so that liquidity is scarce in equilibrium and all firms meet the high liquidity shock. I refer to such a point as an interior equilibrium. Then all firms meet the high liquidity shock, $\zeta = 1$, and the liquidity premium satisfies $q - 1 = C(M_1, K)/M_1$. Thus investment satisfies $I(M_1) = I_1(C(M_1, K)/M_1)$.

Since we have simple expressions for $q(M)$ and $I(q)$, the key determinant of equilibrium is $M = M_1$. We know from Proposition 6 that M_1 satisfies

$$(\rho_H - \rho_0) I_1 \left(\frac{C(M_1, K)}{M_1} \right) = M_1 + \bar{\ell}$$

which allows us to implicitly define a function $M_1(K, \bar{\ell})$.

There is no analytic expression for $M_1(\cdot)$ for general agency costs, so we use the implicit function theorem to derive expressions for the marginal change in equilibrium M_1 from a change in outside liquidity $\bar{\ell}$ or bank capital K. Once we have the derivatives of M with respect to exogenous variables, we can easily compute changes in I and q using $q(M, K) = C(M, K)/K$ and $I(q)$.

4.1 Variation in Outside Liquidity

We first consider the effect of variations in outside liquidity $\bar{\ell}$. Following the approach described in the previous section, we compute the following changes in equilibrium variables in response to a marginal change in $\bar{\ell}$.

Proposition 7. *At an interior equilibrium the effect of a marginal change in $\bar{\ell}$ on equilibrium variables is*

$$\begin{aligned}\frac{dM}{d\bar{\ell}} &= -\left(1+\epsilon/\eta\right)^{-1} \in (-1,0)\\ \frac{dq}{d\bar{\ell}} &= -\left(\frac{C_M - C/M}{M}\right)\left(1+\epsilon/\eta\right)^{-1} < 0\\ \frac{dI}{d\bar{\ell}} &= \left(\rho_H-\rho_0\right)^{-1}\left(1+\eta/\epsilon\right)^{-1} > 1\end{aligned}$$

where $\eta = \left(\frac{C/M}{C_M - C/M}\right)\left(\frac{M}{M+\bar{\ell}}\right)$ is the elasticity of liquidity supply, and $\epsilon = \left(M+\bar{\ell}\right)\frac{C/M}{A}$ is the elasticity of liquidity demand.

Proof. See appendix C. □

The signs given follow from our assumptions that $C_M - C/M > 0$ so that $\eta > 0$. Naturally an increase in the supply of outside liquidity will reduce its equilibrium price, so that $dq/d\bar{\ell} < 0$. This is just the typical result that an increase in supply for a good lowers its price. It is equally intuitive that the lower price of liquidity leads to higher investment $dI/d\bar{\ell} > 0$, since liquidity is an input into production.

The result $dM/d\bar{\ell} \in (-1,0)$ is also quite intuitive. M and $\bar{\ell}$ are both forms of liquidity, and so are substitutes. An increase in the supply of one reduces the demand for the other. But the reduction in credit lines must be less than one-for-one, because lower q induces higher investment from firms. Total demand for liquidity $M+\ell$ is proportional to investment, and so must rise also. This implies $dM/d\bar{\ell} > -1$.

Note that since $\partial q/\partial M = (C_M - C/M)/M$, the expression for $dq/d\bar{\ell}$ is equivalent to $\frac{\partial q}{\partial M}\frac{dM}{d\bar{\ell}}$, which is just an application of the chain rule. Also note that $dI/d\bar{\ell}$ is equal to $(\rho_H-\rho_0)^{-1}(1+\frac{dM}{d\bar{\ell}})$, as we would expect given $(\rho_H-\rho_0)I = M+\bar{\ell}$ in equilibrium.

A fall in outside liquidity is depicted in figure 5. This corresponds to a leftward translation of the liquidity supply curve, which raises the liquidity premium and lowers investment.

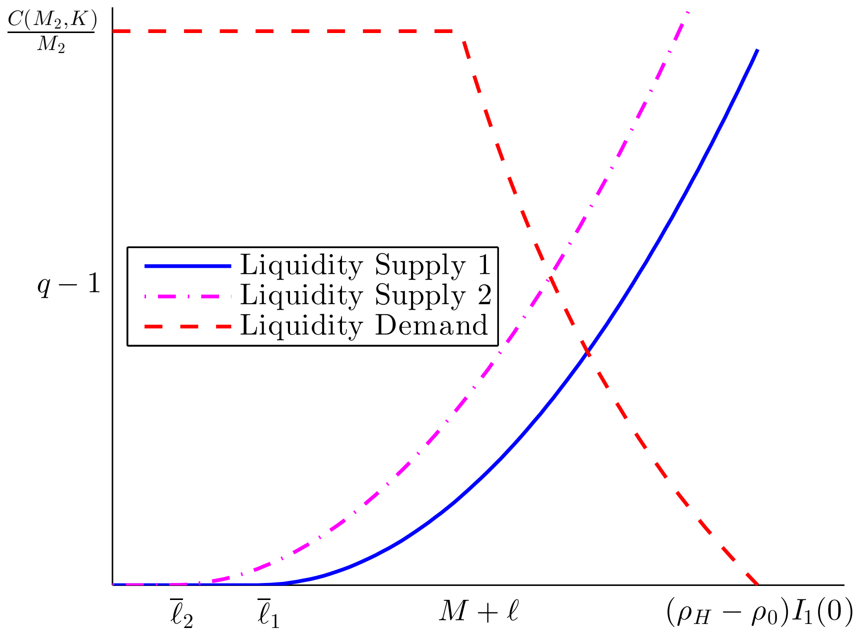

Figure 5: Fall in outside liquidity with banks.

4.2 Variation in Bank Capital

We next consider the effect of a change in bank capital K. We can compute changes in equilibrium variables using the same approach as above. The results are given in Proposition 8.

Proposition 8. *At an interior equilibrium the effects of a marginal change in K on equilibrium variables are*

$$\frac{dM}{dK} = -\left(\frac{M}{C_M - C/M}\right)(1+\eta/\epsilon)^{-1}\frac{C_K}{M} > 0$$
$$\frac{dq}{dK} = (1+\epsilon/\eta)^{-1}\frac{C_K}{M} < 0$$
$$\frac{dI}{dK} = (\rho_H-\rho_0)^{-1}\frac{dM}{dK} > 0$$

where $\eta = \left(\frac{C/M}{C_M - C/M}\right)\left(\frac{M}{M+\bar{\ell}}\right)$ is the elasticity of liquidity supply, and $\epsilon = \left(M+\bar{\ell}\right)\frac{C/M}{A}$ is the elasticity of liquidity demand.

Proof. See appendix C. □

The given signs follow from the assumptions $C_M > C/M$ and $C_K < 0$. Intuitively, an increase in bank capital lowers the profits necessary to induce banks' cooperation,

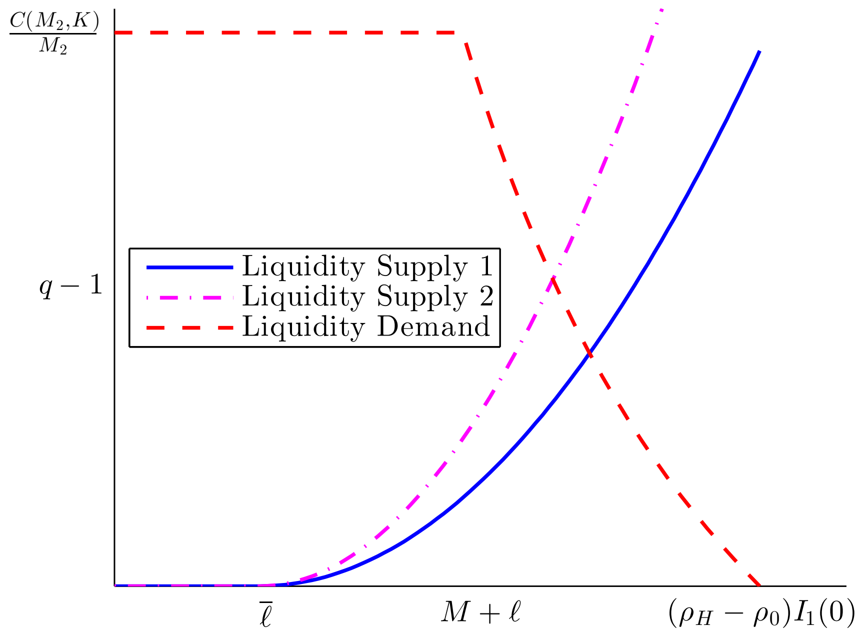

Figure 6: Fall in bank capital.

which raises the effective supply of liquidity. Bank creation of credit lines M increases, the liquidity premium $q-1$ falls, and investment rises.

Comparing these expressions to those for changes in $\bar{\ell}$, we see that the expression for dq/dK is the same as the expression for $dq/d\bar{\ell}$ times the term C_K/M. The latter is the first-order change in q due to the change in bank capital, and thus corresponds to the upward shift in the liquidity supply curve. The reason these expressions are otherwise identical is because the change in quantity due to a rightward shift of a curve is the same as the change in price due to a downward shift of the same curve, as can be easily verified geometrically.

Figure 6 depicts a fall in bank capital. This again shifts the liquidity supply curve to the left. However, rather than being a translation in the curve, it rotates the curve about the point $\bar{\ell}$. The larger the initial M, the larger is the rotation in the supply curve, since larger bank financing implies a larger effect from a fall in bank capital.

5 Welfare

We now turn to the welfare properties of the equilibrium described in section 3. Clearly equilibrium will fail to attain the unconstrained optimal level of investment described in Proposition 1, and so the equilibrium is not Pareto optimal. A more interesting question is whether the equilibrium corresponds to the choice of a *constrained planner*, i.e. a planner with access to a limited set of policy instruments.

Since this notion of optimality is contingent on the instruments available to the constrained planner, the analysis of welfare and policy are closely related. I proceed by first defining a notion of welfare, and comparing it to the equilibrium allocation. I then consider what instruments would enable a planner to implement this allocation. I show that when liquidity is scarce, the equilibrium allocation is generally suboptimal relative to the choices of a constrained planner.

Since utility is linear with no discounting, total welfare is the sum of all agents' consumption. Normalizing zero-investment welfare to 0, total welfare is

$$W = \int_i \left[p(\rho_1 - \rho_L) + \lambda_H^i (1-p)(\rho_1 - \rho_H) - 1 \right] I_i$$

where (I_i, λ_H^i) is the production plan of firm i.

In the following analysis, I assume that the constrained planner maximizes total welfare W. The result may not be a Pareto-improvement relative to equilibrium as the distribution of consumption across agents may change. However, if the planner has access to lump-sum taxes and transfers at the end of period 2, any policy that increases W can be made Pareto improving through suitable uncontingent transfers between agents. Such transfers will not affect agents' behavior given linear utility.

5.1 Unconstrained Optimum

First consider the unconstrained optimal allocation described in Proposition 1. This allocation involves investing all available period 0 resources $I = A + H_0$, and meeting the high liquidity shock $\lambda_H = 1$. Since the maximum level of investment in equilibrium with $\lambda_H = 1$ is $I = A/\chi_1 < A + H_0$, the equilibrium is not Pareto optimal.

A planner that has access to lump-sum taxes and transfers in all periods can implement the unconstrained optimum by transferring sufficient funds from households to firms in periods 0 and 1, as described in Proposition 9.

Proposition 9 (Planner's Problem with Unlimited Transfers)**.** *A planner that has access to unlimited lump-sum taxes and transfers can achieve the allocation described in Proposition 1 by transferring H_0 from households to firms in period 0, and transferring $(\rho_H - \rho_0)(A + H_0)$ from households to firms that experience a high liquidity shock in period 1.*

Proof. See appendix D. □

Since the only constraint on investing all available period 0 funds is limited access to financing in periods 0 and 1 arising from limited pledgeability, a planner can achieve the optimal production plan by simply transferring the necessary funds to firms. Intuitively, if a planner can identify investment opportunities that cannot be financed by the private sector via credit markets, and has access to nondistortionary tax instruments, then the planner can improve welfare by transferring funds directly to those agents with investment opportunities. Thus the planner wholly supplants private credit markets. However, such policies are rarely observed in practice since it is rare that governments hold the necessary informational advantage over the private sector.

5.2 Liquidity-Constrained Optimum

In Sections 2 and 3 I defined the constrained optimum as the equilibrium when liquidity is abundant, i.e. when $\bar{\ell}$ is sufficiently large that $q = 1$. For the discussion of welfare, another notion of constrained optimality is useful: the optimal allocation given fixed liquidity supply $\bar{\ell}$ and K, and subject to agents' participation and incentive compatability constraints. To distinguish this from the previously defined notion of constrained optimum, I refer to this as the *liquidity-constrained optimum*.

Here I define an allocation as $\{I_i, \ell_i, \pi_i, M_i\}_i$ for firms $i \in [0,1]$, together with price of liquidity q. The participation constraint for suppliers of outside liquidity is $q \geq 1$. Investors receive compensation out of the pledgeable funds of firms, so investment satisfies

$$I_i \leq \frac{A - \pi_i - (q-1)\ell_i}{1 - p(\rho_0 - \rho_L) - \lambda_H^i(1-p)(\rho_0 - \rho_H)} \tag{5.1}$$

Each firm that meets the high shock must have sufficient liquidity to meet the shock

$$\lambda_H^i I_i(\rho_H - \rho_0) \leq M_i + \ell_i \tag{5.2}$$

Banks must receive sufficient compensation for providing liquidity

$$\int_i \pi_i \geq C\left(\int_i M_i, K\right) \tag{5.3}$$

Finally, firms will not pay for liquidity if they are not using it, so $\pi_i = 0$ if $\lambda_H^i = 0$, and outside liquidity use is limited by the available supply $\int_i \ell_i \leq \bar{\ell}$.

Definition 2. The *liquidity constrained optimum* (LCO) is the solution to

$$\max_{I_i,\lambda_H^i,M_i,\ell_i,\pi_i,q} \int_i \left[p(\rho_1-\rho_L)+\lambda_H^i(1-p)(\rho_1-\rho_H)-1\right] I_i$$
$$\text{s.t.} \quad (5.1), (5.2), (5.3), \int_i \ell_i \leq \bar{\ell}, q \geq 1$$

Proposition 10 (Liquidity Constrained Optimum)**.** *Let* $\chi_1 = 1-\rho_0+p\rho_L+(1-p)\rho_H$ *and* $\chi_0 = 1-p(\rho_0-\rho_L)$. *Let* ζ *designate the fraction of firms that meet the high liquidity shock. Let* $R_1 = \rho_1 - p\rho_L - (1-p)\rho_H - 1$ *and* $R_0 = p(\rho_1-\rho_L)-1$. *Let* $I(\zeta)$ *be defined by* $\chi_1 I = A - C\left(\max(\zeta(\rho_H-\rho_0)I-\bar{\ell},0),K\right)/\zeta$, *and let* $\hat{\ell}$ *be defined by* $R_1\chi_0 = R_0\left[\chi_1+(\rho_H-\rho_0)C_1\left((\rho_H-\rho_0)I(1)-\hat{\ell},K\right)\right]$. *Then the liquidity constrained optimum is defined as follows:*

(i) If $\bar{\ell} \geq (\rho_H-\rho_0)\frac{A}{\chi_1}$, *then all firms choose* $\lambda_H = 1$, $I = \frac{A}{\chi_1}$, $\ell = (\rho_H-\rho_0)\frac{A}{\chi_1}$, *and* $M_i = \pi_i = 0$.

(ii) If $\hat{\ell} \leq \bar{\ell} < (\rho_H-\rho_0)\frac{A}{\chi_1}$, *then all firms choose* $\lambda_H = 1$, $I = I(1)$, $\ell = \bar{\ell}$, $M = (\rho_H-\rho_0)I-\bar{\ell}$, *and* $\pi = C(M,K)$.

(iii) If $\bar{\ell} < \hat{\ell}$, *then a fraction* ζ *of firms choose* $\lambda_H = 1$, *where* ζ *is implicitly defined by*

$$R_1\chi_0 = R_0\left[\chi_1+(\rho_H-\rho_0)C_1\left(\zeta(\rho_H-\rho_0)I(\zeta)-\bar{\ell},K\right)\right]$$

These firms choose $I = I(\zeta)$, $\ell = \bar{\ell}/\zeta$, $M = (\rho_H-\rho_0)I(\zeta)-\ell$, *and* $\pi = C(\zeta M,K)/\zeta$. *The remaining fraction* $1-\zeta$ *choose* $\lambda_H = 0$, $I = A/\chi_0$, $\ell = M = \pi = 0$.

$q = 1$ *in all cases.*

Proof. See appendix D. □

The differences between the equilibrium and the liquidity constrained optimum arise because firms do not consider the effects of their demand for liquidity on the prices of liquidity q and π, whereas the planner internalizes these pecuniary effects. Since payments for liquidity reduce pledgeable funds, a higher price of liquidity reduces possible investment.

When liquidity is scarce in equilibrium, firms bid up the price of outside liquidity to some $q > 1$. Since higher q transfers period 0 funds from firms to households who hold outside liquidity, it inefficiently decreases investment. Since the supply of outside liquidity is inelastic, there is no gain from increasing its price. Thus the constrained planner sets the price of ℓ as low as possible, resulting in higher investment.

We can see this by comparing the expressions for I in the cases of equilibrium and the liquidity constrained optimum. For simplicity consider the case that $\zeta = 1$, so that all firms meet the liquidity shocks. Then equilibrium investment satisfies

$$I = \frac{A - C(M,K) - \frac{C(M,K)}{M} \cdot \bar{\ell}}{\chi_1}$$

where $M = I/(\rho_H - \rho_0) - \bar{\ell}$. By comparison, the liquidity-constrained level of investment satisfies

$$I = \frac{A - C\,(M,K)}{\chi_1}$$

where again $M = I/(\rho_H - \rho_0) - \bar{\ell}$.

The difference is that in equilibrium, pledgeable funds are reduced by a further amount $C/M \cdot \bar{\ell}$. This occurs because arbitrage pushes up the price of outside liquidity to equal the cost of credit lines provided by banks. Since outside liquidity is in fixed supply, this increase in price does not raise the supply of outside liquidity. Instead, owners of outside liquidity enjoy an excess return that comes at a steep social cost, since it reduces pledgeable funds and therefore investment. A planner would prefer to push down the price of liquid assets in order to increase investment, which implies a wedge between the price of outside liquidity and the premium paid to financial intermediaries. The former fulfills no social purpose, while the latter is necessary to overcome the agency costs of banks.

Implementation. Given the analysis above, a constrained planner should try to lower the price of outside liquidity in order to increase welfare. The simplest way to do this is to put a price ceiling on the price of outside liquidity at $q = 1$. Then firms would first buy outside liquidity until it was sold out, and then substitute to the (more expensive) bank credit lines.

6 Public Liquidity Provision

The previous sections assumed that the supply of liquid assets in the economy is fixed. However, many government policies directly affect this supply. When central banks engage in conventional monetary policy, they do so by buying and selling assets of varying liquidity. Likewise, when governments issue new bonds, they increase the economy's store of liquid assets. We can capture these activities in our model as the issuance of government bonds that are perfect substitutes for outside liquidity.

Throughout this section, we will focus on the interior equilibrium where both liquidity shocks are met and the liquidity premium is strictly positive.

6.1 Public Bond Issuance

Suppose that the government issues x bonds with a face value of 1 in period 0. The funds from the sale of these bonds are returned to households via a lump sum transfer and consumed immediately. In period 2, the government levies a tax on the households in order to raise funds that it uses to repay the bonds.

Since government bonds are perfect substitutes for outside liquid assets $\bar{\ell}$, government bonds will sell at the same equilibrium price q. From the perspective of firms and households, it is as though the stock of outside liquidity had increased from $\bar{\ell}$ to $\bar{\ell} + x$, and so equilibrium will be exactly the same as given in Proposition 6, except that $\bar{\ell}$ is replaced by $\bar{\ell} + x$. Likewise the comparative statics of real variables with respect to changes in government debt x will be as given in Proposition 7, with $\bar{\ell}$ replaced by $\bar{\ell} + x$, and $d\bar{\ell}$ replaced by dx.

By assumption, the government has perfect credibility, and therefore is able to commit to repaying its debt. Since the government can raise taxes, it can promise a sufficiently large quantity of pledgeable funds in period 1 to cover any potential liquidity shock. By Proposition 6, there is a level of outside liquidity that achieves the constrained optimum, and so the government can issue bonds in order to achieve the constrained optimum level of investment. If there were no costs to taxation, then this would be the optimal policy, and the government could achieve the constrained optimum. However, I assume that the government can only raise funds using a distortionary tax, although it can disburse funds to households in a lump-sum transfer. Suppose that the deadweight loss from raising x funds is given by the function $D(x)$, which I assume is increasing and convex in x, and satisfies $D'(0) = 0$ and $D'(x) \to \infty$ as $x \to \infty$.[19]

6.2 Results with General Agency Costs

The total social surplus if both shocks are met is

$$RI - D(x)$$

[19] I assume here that the government must raise funds by taxing activity in some unmodeled market.

where $R = \rho_1 - p\rho_L - (1-p)\rho_H - 1$, since all other aspects of the allocation represent transfers.[20] Then at an interior solution the necessary condition for optimality is

$$R\frac{dI}{dx} = D'(x) \tag{6.1}$$

which will imply a unique optimum if $\frac{d^2I}{dx^2} < 0$ for all x. However, given the expression for $\frac{dI}{d\ell}$ derived previously in Proposition 7, the sign of $\frac{d^2I}{dx^2}$ will depend on how the ratio of elasticities η/ϵ vary as outside liquidity is changed. Thus the sign will be ambiguous, and without further assumptions we cannot say whether $\frac{dI}{dx}$ is increasing or decreasing in x, and we cannot give an expression that uniquely defines optimal policy. We can, however, say a few things about optimal policy in general.

Proposition 11. *Suppose that in the absence of government policy the economy is initially at an interior equilibrium. Then*

(i) The optimal supply of government liquid assets x^* will be positive.

(ii) If optimal x^* implies that the economy is still at an interior equilibrium, and if the current choice of x^* is unique, then a marginal change in $\bar{\ell}$ or K will shift the optimal point according to

$$\frac{dx^*}{d\bar{\ell}} = -\frac{R\frac{\partial^2 I}{\partial x^2}}{R\frac{\partial^2 I}{\partial x^2} - D''(x)}$$

$$\frac{dx^*}{dK} = -\frac{R\frac{\partial^2 I}{\partial x \partial K}}{R\frac{\partial^2 I}{\partial x^2} - D''(x)}$$

(iii) At an interior optimal point x^*, we have $R\frac{\partial^2 I}{\partial x^2} < D''(x)$.

(iv) We have $\frac{dx}{d\bar{\ell}} > 0 \iff \frac{\partial^2 I}{\partial x^2} > 0$ and $\frac{dx}{dK} > 0 \iff \frac{\partial^2 I}{\partial x \partial K} > 0$.

Proof. See appendix E. □

The first result in Proposition 11 is that if the economy is at an interior equilibrium in the absence of any policy, it will always be optimal to choose $x^* > 0$. Thus when liquidity is scarce it will in general be optimal for the government to issue bonds solely for their value in providing liquidity, even though taxation is distortionary. The intuition for this result is that at an interior solution with a positive liquidity premium there is a positive marginal value of issuing bonds because they serve as liquid assets. Since the marginal

[20] This is with the no-production case normalized to 0. Additional liquidity has no value unless both shocks are met, so we will restrict attention to this case.

cost of taxation is zero at $x = 0$, the marginal value of issuing bonds at $x = 0$ is greater than the marginal cost, and therefore it will be always be optimal to issue some positive quantity of bonds.[21]

This intuition is depicted in Figure 7. The line that passes through the origin is the marginal cost of raising x in funds via taxation, and the other line is the marginal value of increasing the supply of liquid assets $R\frac{dI}{dx}$. Moreover, there is a point $\bar{x} = \frac{A(\rho_H - \rho_0)}{\chi_1} - \bar{\ell}$, such that if the government chooses $x \geq \bar{x}$ the economy achieves the constrained optimum, and the marginal value of increasing x above $\bar{x}$ drops to zero. Therefore there will be some point at which the two curves intersect, and this will be at some $x > 0$.

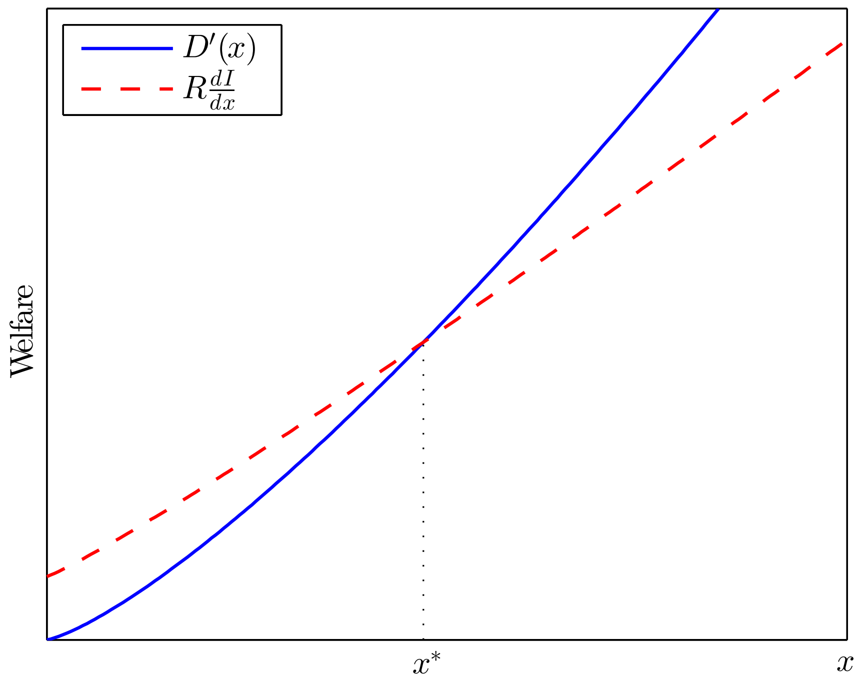

Figure 7: Optimal policy.

Results (ii) - (iv) in Proposition 11 have a similarly straightforward intuition. Suppose that there is an interior unique solution at the point x^*. Then at x^* the marginal cost and marginal benefit curves intersect, as depicted in Figure 7. Since x^* is a local maximum, the marginal benefit curve crosses the marginal cost curve from above, as claimed in statement (iii). A change in $\bar{\ell}$ or K will not shift the marginal cost curve. Since the equation for the marginal benefit curve is $R\frac{dI}{dx}$, an increase in $\bar{\ell}$ will shift the marginal benefit curve upwards if $\frac{\partial^2 I}{\partial x^2} > 0$, which will increase the point of intersection x^*. Similarly if $\frac{\partial^2 I}{\partial x \partial K} > 0$,

[21]This assumes that the cost of the first dollar raised is zero. If $D'(0) > 0$, as will be the case if there are distortionary taxes already in place, optimal public debt issuance may be zero.

an increase in K will shift the marginal benefit curve upwards, which will again increase the point of intersection x^*, and the reverse. Such a shift is depicted in Figure 8.

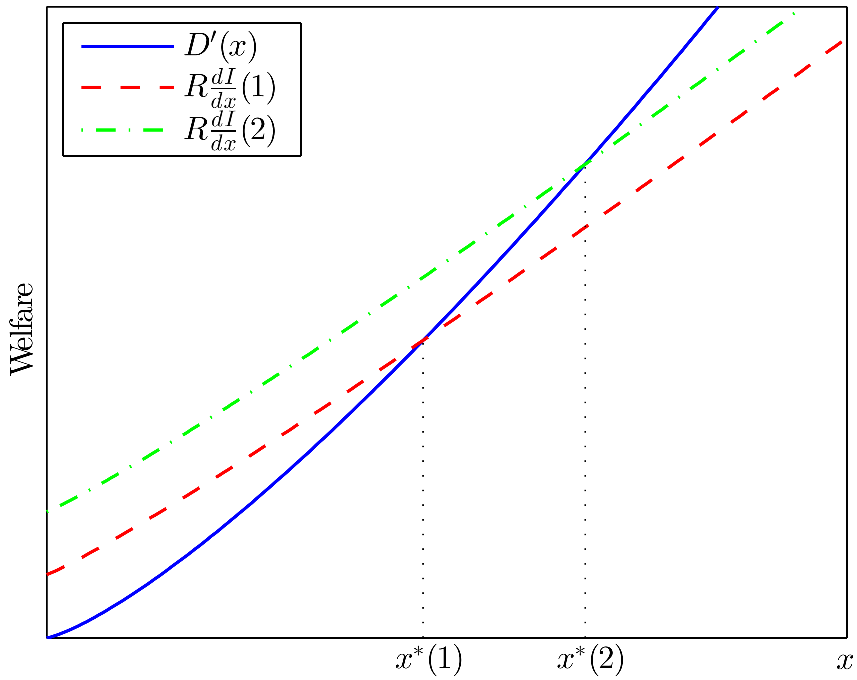

Figure 8: Optimal policy after increase in outside liquidity.

6.3 Optimal Policy and the Elasticity of Bank Liquidity Supply

The previous section made no assumptions about the bank agency cost functions beyond what we have already assumed, and so was not able to say very much about either the level of optimal bond supply x^*, or how x^* varies with private outside liquidity or bank capital. However, under a few reasonable assumptions about the agency cost function, we can say a lot more.

For this section I change the terminology slightly for ease of exposition. We continue to only consider the case that firms meet both liquidity shocks. Let the total liquidity used be L, so that $L = M + x + \bar{\ell}$; let the liquidity premium be $\vartheta = q - 1$; let total outside liquidity, including both government bonds and private outside liquidity, be $z = x + \bar{\ell}$; let $\vartheta(M, K) = C(M, K)/M$ be the *average* agency cost function; finally, let $\phi = (\rho_H - \rho_0)^{-1}$ be the liquidity multiplier, so that $I = \phi L$.

Using this terminology, the elasticity of liquidity demand is

$$\epsilon = \frac{\vartheta}{\vartheta + \phi\chi} = \frac{L\vartheta}{A}$$

and the elasticity of liquidity supply is

$$\eta = \frac{\vartheta}{\vartheta_M L} = \eta_M \left(\frac{L - z}{L} \right)$$

where $\eta_M = \frac{\vartheta}{\vartheta_M M}$ is the elasticity of the supply of bank credit lines. This decomposition reflects that the economy's liquidity is the sum of two components: outside liquidity that is supplied inelastically, and bank liquidity that is supplied with elastiicty η_M. Thus total elasticity of liquidity supply is a function of η_M, and the fraction of total liquidity that is supplied by banks.

Using these expressions, we can prove the following about the optimal supply of government bonds x^*:

Proposition 12. *Suppose that under optimal government bond issuance x^* the economy is at an interior equilibrium. Then*

(i) *x^* is decreasing in the ratio of the elasticity of liquidity supply to the elasticity of liquidity demand η/ϵ.*

(ii) *x^* is* increasing *in private outside liquidity $\bar{\ell}$ iff*

$$\frac{\frac{d}{dz}(\eta_M)}{\eta_M} + \frac{1 - 2\epsilon - \eta_M}{L(\epsilon + \eta)} < 0 \tag{6.2}$$

(iii) *A sufficient condition for (ii) to hold is that the supply of credit lines by bank be isoelastic with elasticity $\eta_M \geq 1$.*

(iv) *x^* is* decreasing *in bank capital K iff*

$$\frac{L_{zz}}{L_z} > \frac{d}{dM} \left(\frac{\vartheta_M}{\vartheta_K} \right) \cdot \frac{\vartheta_M}{\vartheta_K} M_z \tag{6.3}$$

Proof. See appendix E. □

These results suggest that for a reasonable parameterization of the bank agency cost, the optimal supply of government bonds is *increasing* in private outside liquidity $\bar{\ell}$, and *decreasing* in bank equity K. In other words, the government should provide additional public liquidity when private liquidity from banks decreases, but should provide *less* liquidity when private outside liquidity decreases. I discuss the intuition for each of these results below. The key mechanism is the response of bank liquidity supply to government liquidity provision, since government liquidity crowds out (elastically supplied) bank liquidity, but does not crowd out (inelastically supplied) private outside liquidity.

Result (i) is immediate from the expression for L_x:

$$L_x = 1 + M_x = \frac{\epsilon/\eta}{1+\epsilon/\eta}$$

together with the optimality expression for government debt (6.1). Intuitively, the ratio η/ϵ tells us to what extent an increase in outside liquidity will be offset by a reduction in bank liquidity. η tells us how much bank liquidity will fall for a given fall in the price of liquidity ϑ, and $1/\epsilon$ tells us how much the price of liquidity will fall to absorb a given increase in liquidity. If η is large, issuance of government liquidity will cause a large reduction in bank liquidity, and so the benefits of public liquidity provision are attenuated. Likewise, if ϵ is small, demand for liquidity is very inelastic, which suggests a large price swing from public liquidity provision, which will prompt a greater crowding out of bank liquidity. Figure 9 depicts the optimal supply of government liquidity x^* for high and low η/ϵ.

Result (ii) and (iii) give conditions under which x^* decreases given a decrease in private outside liquidity $\bar{\ell}$. This will hold as long as bank liquidity is reasonably elastic, and its elasticity is not falling too rapidly in the price of liquidity. In particular, this will hold if bank liquidity is isoelastic, with an elasticity of at least 1. This result might seem somewhat surprising — one might intuitively expect that the government should provide outside liquidity when it is scarce. However, the normal intuition rests on the idea of decreasing marginal returns on investment, with liquidity a scarce input. That does not hold here, since we assumed R was fixed. Instead, an increase in private outside liquidity $\bar{\ell}$ makes the overall supply of liquidity more inelastic, which decreases crowding out from government supply of liquidity. This is what drives the sign.

Result (iv) gives a condition under which x^* *increases* given a decrease in bank capital K. This is close to what we would expect intuitively, but in this case it is again driven by changes in the elasticity of liquidity supply. The condition that establishes this result requires that the cross elasticity of the agency cost function not be too great. If this failed, the increase in K might end up *decreasing* the elasticity of liquidity supply: although it would increase the share of (elastic) bank liquidity in the total, it would simultaneously reduce the elasticity of bank liquidity supply enough that the net result would be to decrease the

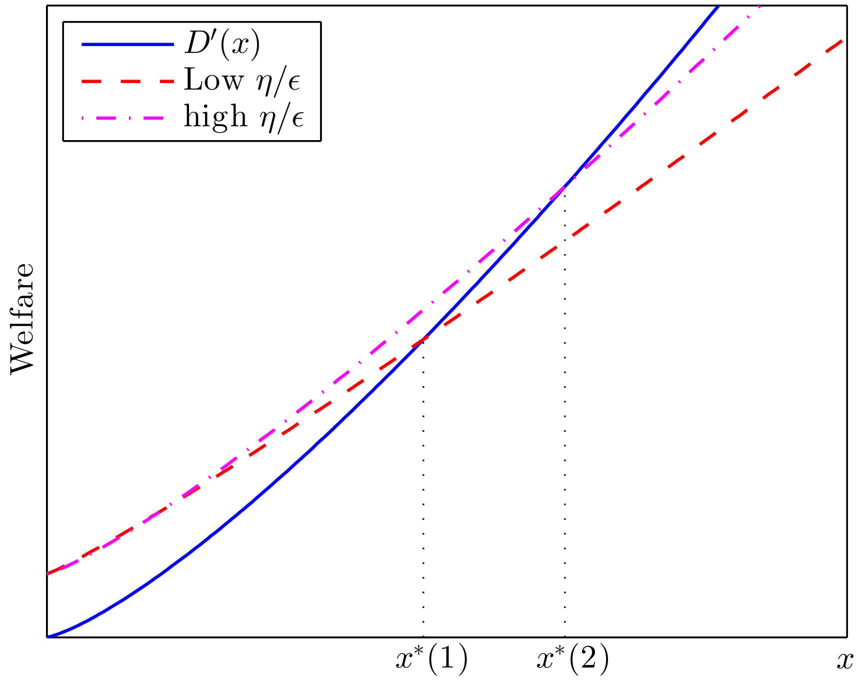

Figure 9: Optimal policy for high and low elasticity ratios.

elasticity of liquidity supply overall.

7 Conclusion

I constructed a model in which variations in bank capital and the stock of liquid assets affect investment and production through a cost of liquidity channel. I show that when the stock of liquid assets falls below a critical threshold, equilibrium investment will be below the constrained optimal level. The level of investment will depend on the liquidity premium, which in equilibrium will equal the average agency costs of banks. The government can provide liquidity by issuing public liabilities such as bonds, and if liquidity is scarce it will in general be optimal to do so. The optimal supply of public liquidity is decreasing in bank capital, providing a rationale for countercyclical public liquidity provision. Importantly, this does not refer to conventional monetary policy or special credit facilities, but to the issuance of new liquid assets such as government bonds, which are useful to the productive sector as a complement to illiquid capital. Government debt issuance can "crowd in" investment when liquid assets are scarce, providing a new justification for countercyclical budget deficits.

References

Acharya, V.V. and P. Schnabl. 2010. "Do Global Banks Spread Global Imbalances? Asset-Backed Commercial Paper during the Financial Crisis of 2007–09." *IMF Economic Review* 58 (1):37–73.

Adrian, Tobias and Hyun Song Shin. 2010. "Liquidity and leverage." *Journal of financial intermediation* 19 (3):418–437.

Akerlof, George A. 1970. "The Market for "Lemons": Quality Uncertainty and the Market Mechanism." *The Quarterly Journal of Economics* :488–500.

Bernanke, B., C. Bertaut, L. DeMarco, and S. Kamin. 2011. "International Capital Flows and the Return to Safe Assets in the United States, 2003-2007." *FRB International Finance Discussion Paper No. 1014* .

Bernanke, B. and M. Gertler. 1989. "Agency Costs, Net Worth, and Business Fluctuations." *The American Economic Review* 79 (1):14–31.

Bernanke, B., M. Gertler, and S. Gilchrist. 1999. "The financial accelerator in a quantitative business cycle framework." *Handbook of Macroeconomics* 1:1341–1393.

Brunnermeier, Markus K and Yuliy Sannikov. 2016. "The I Theory of Money." *NBER Working Paper* w22533.

Brunnermeier, M.K. 2009. "Deciphering the Liquidity and Credit Crunch 2007-2008." *The Journal of Economic Perspectives* 23 (1):77–100.

Caballero, R.J. 2010. "The "Other" Imbalance and the Financial Crisis." *NBER Working Paper* w15636.

Calomiris, C.W. and C.M. Kahn. 1991. "The Role of Demandable Debt in Structuring Optimal Banking Arrangements." *The American Economic Review* :497–513.

Dang, Tri Vi, Gary Gorton, Bengt Holmström, and Guillermo Ordonez. 2017. "Banks as secret keepers." *The American Economic Review* 107 (4):1005–1029.

DeMarzo, Peter and Darrell Duffie. 1999. "A Liquidity-Based Model of Security Design." *Econometrica* 67 (1):65–99.

Diamond, D.W. and P.H. Dybvig. 1983. "Bank Runs, Deposit Insurance, and Liquidity." *The Journal of Political Economy* 91 (3):401–419.

Diamond, D.W. and R.G. Rajan. 2001. "Liquidity Risk, Liquidity Creation, and Financial Fragility: A Theory of Banking." *Journal of Political Economy* 109 (2).

Farhi, E. and J. Tirole. 2012. "Bubbly Liquidity." *Review of Economic Studies* 79 (2):678–706.

Gertler, Mark and Peter Karadi. 2011. "A model of unconventional monetary policy." *Journal of monetary Economics* 58 (1):17–34.

Gorton, Gary and George Pennacchi. 1990. "Financial Intermediaries and Liquidity Creation." *The Journal of Finance* 45 (1):49–71.

Greenwood, R., S.G. Hanson, and J.C. Stein. 2010. "A Comparative-Advantage Approach to Government Debt Maturity." .

Hölmstrom, Bengt and Jean Tirole. 1998. "Private and Public Supply of Liquidity." *Journal of Political Economy* 106 (1):1–40.

———. 2011. *Inside and Outside Liquidity*. MIT press.

Kashyap, A.K. and J.C. Stein. 2000. "What Do a Million Observations on Banks Say About the Transmission of Monetary Policy?" *American Economic Review* :407–428.

Kiyotaki, N. and J. Moore. 1997. "Credit Cycles." *The Journal of Political Economy* 105 (2):211–248.

Kiyotaki, Nobuhiro and John Moore. 2012. "Liquidity, Business Cycles, and Monetary Policy." *NBER Working Paper* w17934.

Krishnamurthy, Arvind and Annette Vissing-Jorgensen. 2012. "The Aggregate Demand for Treasury Debt." *Journal of Political Economy* 120 (2):233 – 267.

Mattesini, F., C. Monnet, and R. Wright. 2009. "Banking: a Mechanism Design Approach." *Working Paper* .

Pozsar, Z., T. Adrian, A. Ashcraft, and H. Boesky. 2013. "Shadow Banking." *Federal Reserve Bank of New York Economic Policy Review* 19 (2).

Pozsar, Zoltan. 2013. "Institutional Cash Pools and the Triffin Dilemma of the US Banking System." *Financial Markets, Institutions & Instruments* 22 (5):283–318.

Rösch, Christoph G and Christoph Kaserer. 2013. "Market liquidity in the financial crisis: The role of liquidity commonality and flight-to-quality." *Journal of Banking & Finance* 37 (7):2284–2302.

Samuelson, P.A. 1958. "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money." *The Journal of Political Economy* 66 (6):467–482.

Stein, J.C. 2012. "Monetary Policy as Financial Stability Regulation." *The Quarterly Journal of Economics* 127 (1):57–95.

Sufi, Amir. 2009. "Bank Lines of Credit in Corporate Finance: An Empirical Analysis." *Review of Financial Studies* 22 (3):1057–1088.

Tirole, Jean. 2011. "Illiquidity and All Its Friends." *Journal of Economic Literature* 49 (2):287–325.

Woodford, M. 1990. "Public Debt as Private Liquidity." *The American Economic Review* 80 (2):382–388.

A Online Appendix: Proofs from section 2

Proof of Proposition 1. We can solve (2.4) using the Lagrangian

$$L = \sum_{i=0}^{2} C_i + \mu_0 \left[A + H_0 - C_0 - I\right] + \mu_1 \left[H_1 - C_1 - p\lambda_L\rho_L I - (1-p)\lambda_H\rho_H I\right]$$
$$+ \mu_2 \left[H_2 + p\lambda_L\rho_1 I + (1-p)\lambda_H\rho_1 I + l - C_2\right] + \sum_{i=0}^{2} \nu_i C_i + \nu_3 I$$

The first-order condition with respect to C_i is

$$\mu_i = 1 + \nu_i$$

which implies $\mu_i > 0$, and so the period budget constraints hold with equality. Substituting them directly into the objective function, we obtain the new problem

$$\max_{\lambda_L,\lambda_H,I} \left\{A + H_0 + H_1 + H_2 + p\lambda_L\left(\rho_1 - \rho_L\right)I + (1-p)\lambda_H\left(\rho_1 - \rho_H\right)I - I\right\} \tag{A.1}$$
$$\text{s.t.} \quad H \geq I - A \tag{A.2}$$
$$H \geq p\lambda_L\rho_L I + (1-p)\lambda_H\rho_H I \tag{A.3}$$
$$I \geq 0$$

where $\lambda_s \in \{0, 1\}$ for $s \in \{L, H\}$. The constraints (A.2) and (A.3) are the non-negativity constraints on C_0 and C_1 respectively. The Lagrangian of (A.1) is

$$L = A + H_0 + H_1 + H_2 + p\lambda_L\left(\rho_1 - \rho_L\right)I + (1-p)\lambda_H\left(\rho_1 - \rho_H\right)I - I$$
$$+ \mu_1\left[A + H - I\right] + \mu_2\left[H - p\lambda_L\rho_L I - (1-p)\lambda_H\rho_H I\right] + \mu_3 I \tag{A.4}$$

The first-order condition of (A.4) with respect to I is

$$p\lambda_L\left(\rho_1 - \rho_L\right) + (1-p)\lambda_H\left(\rho_1 - \rho_H\right) - 1 = \mu_2\left[p\lambda_L\rho_L + (1-p)\lambda_H\rho_H\right] + \mu_1 - \mu_3$$

By assumption, there will always be sufficient funds in period 1 to meet liquidity shocks, meaning $H_1 \geq p\lambda_L\rho_L I + (1-p)\lambda_H\rho_H I$, and so we have $\mu_2 = 0$. Thus the first-order condition reduces to

$$p\lambda_L\left(\rho_1 - \rho_L\right) + (1-p)\lambda_H\left(\rho_1 - \rho_H\right) - 1 = \mu_1 - \mu_3$$

The left-hand side $p\lambda_L(\rho_1 - \rho_L) + (1-p)\lambda_H(\rho_1 - \rho_H) - 1$ is the net return on investment. If the net return is positive, then $\mu_1 > 0$, the non-negativity constraint on C_0 binds, and the economy invests all available funds in period 0. If the net return is negative, then $\mu_3 > 0$, the non-negativity constraint on I binds, and the economy does not invest anything. We can express this investment rule as

$$I = \begin{cases} A + H_0 & p\lambda_L(\rho_1 - \rho_L) + (1-p)\lambda_H(\rho_1 - \rho_H) > 1 \\ [0, A + H_0] & p\lambda_L(\rho_1 - \rho_L) + (1-p)\lambda_H(\rho_1 - \rho_H) = 1 \\ 0 & p\lambda_L(\rho_1 - \rho_L) + (1-p)\lambda_H(\rho_1 - \rho_H) < 1 \end{cases}$$

Now we just need to determine the optimal choices of λ_L and λ_H. Increasing λ_L and λ_H from 0 to 1 results in increases in the objective function of $p\lambda_L(\rho_1 - \rho_L)I$ and $(1-p)\lambda_H(\rho_1 - \rho_H)I$ respectively. Given our assumption that $\rho_1 > \rho_H > \rho_L$, both of these terms are positive for $I > 0$. Therefore it is optimal to choose $\lambda_L = 1$ and $\lambda_H = 1$. Since the return from a project is greater than the additional cost of bringing the project to completion after the liquidity shock is realized, the optimal continuation policy is to continue in all cases.

We can now define the unconstrained optimum. The solution is $\{C_0, C_1, C_2, I, \lambda\}$ with

$$\begin{aligned} C_0 &= A + H_0 - I \\ C_1 &= H_1 - p\rho_L I - (1-p)\rho_H I \\ C_2 &= H_2 + p\rho_1 I + (1-p)\rho_1 I \\ \lambda_L &= 1 \\ \lambda_H &= 1 \\ I &= \begin{cases} A + H_0 & R \geq 0 \\ 0 & R < 0 \end{cases} \end{aligned}$$

where $R = p(\rho_1 - \rho_L) + (1-p)(\rho_1 - \rho_H) - 1$. □

Proof of Lemma 1. We need to show that it is always worthwhile to increase R_s^I in order to increase I as long as the constraint on pledgeable income in the s state does not bind. If this is true, then it follows that in any equilibrium with positive external financing, limited pledgeability binds, meaning that firms receive exactly the amount necessary for them to cooperate, and no more.

The first step is to argue that the period 1 investors will be paid exactly the funds necessary to finance meeting the liquidity shocks, and no more. This should not be controversial, since there is no other benefit to increasing payments to period 1 investors. To show this, we take the Lagrangian of the problem

$$\begin{aligned}
L = & \, p\left(\lambda_L\rho_1 I - R_L^I - R_L^1 + \ell\right) + (1-p)\left(\lambda_H\rho_1 I - R_H^I - R_H^1 + \ell\right) \\
& + \mu_1\left[R_L^1 - \lambda_L\rho_L I\right] + \mu_2\left[R_H^1 - \lambda_H\rho_H I\right] \\
& + \mu_3\left[pR_L^I + (1-p)R_H^I - I - q\ell + A\right] \\
& + \mu_4\left[\lambda_H\rho_0 I + \ell - R_H^I - R_H^1\right] + \mu_5\left[\lambda_L\rho_0 I + \ell - R_L^I - R_L^1\right]
\end{aligned}$$

and differentiate with respect to R_L^1 and R_H^1. This yields conditions

$$\begin{aligned}
\mu_1 &\leq \mu_4 + p \\
\mu_2 &\leq \mu_5 + 1 - p
\end{aligned}$$

which hold with equality if the corresponding $R_s^1 > 0$. We also have $R_L^1 \geq \lambda_L\rho_L I$ and $R_H^1 \geq \lambda_H\rho_H I$, which indicates that if $I > 0$ and $\lambda_s > 0$, we have $R_s^1 > 0$. So we can conclude that in fact we have $R_s^1 = \lambda_s\rho_s I$, for $s \in \{L, H\}$. Substituting these terms directly, we can rewrite the Lagrangian as

$$\begin{aligned}
L = & \, p\left(\lambda_L\rho_1 I - R_L^I - \lambda_L\rho_L I + l\right) + (1-p)\left(\lambda_H\rho_1 I - R_H^I - \lambda_H\rho_H I + l\right) \\
& + \mu_3\left[pR_L^I + (1-p)R_H^I - I - ql + A\right] \\
& + \mu_4\left[\lambda_H\rho_0 I + l - \lambda_H\rho_H I - R_H^1\right] + \mu_5\left[\lambda_L\rho_0 I + l - R_L^I - \lambda_L\rho_L I\right]
\end{aligned}$$

Now we derive conditions for optimal λ_s. From the Lagrangian, we find that the marginal values of increasing λ_L and λ_H are respectively

$$\begin{aligned}
p\left(\rho_1 - \rho_L\right) I + \mu_5\left(\rho_0 - \rho_L\right) I \\
(1-p)\left(\rho_1 - \rho_H\right) I + \mu_5\left(\rho_0 - \rho_H\right) I
\end{aligned}$$

The optimal choice of λ_s is 1 if the marginal value is positive, and 0 if the marginal value is negative. Since by assumption we have $\rho_1 > \rho_L$ and $\rho_0 > \rho_L$, the first condition is strictly positive, which means that we have $\lambda_L = 1$. The sign of the second is ambiguous since $\rho_0 < \rho_H$, and so the value of λ_H is not clear.

The next step is to observe that a pledgeability constraint will bind if (1) that shock is met, and (2) the multiplier on investment is greater than 1, i.e. $\mu_3 > 1$. This is logical because the direct cost of an increase in R_s^I is 1, so if the value of increasing investment is greater than 1, the optimal solution will be to increase R_s^I until some other constraint binds. This follows from the first-order conditions with respect to R_L^I and R_H^I, which are respectively

$$\mu_5 \geq p\left(\mu_3 - 1\right)$$
$$\mu_4 \geq (1-p)\left(\mu_3 - 1\right)$$

These show that the pledgeability constraints bind if and only if $\mu_3 > 1$.

Now we need only show that $\mu_3 > 1$ to establish the claim. To show this, we differentiate the Lagrangian with respect to I, which yield

$$\mu_3 \leq \left[p\left(\rho_1 - \rho_L\right) + \mu_5\left(\rho_0 - \rho_L\right)\right]\lambda_L + \left[(1-p)\left(\rho_1 - \rho_H\right) + \mu_4\left(\rho_0 - \rho_H\right)\right]\lambda_H$$

which holds with equality when $I > 0$. Since we have $\lambda_H = 0$ whenever $(1-p)\left(\rho_1 - \rho_H\right) + \mu_4\left(\rho_0 - \rho_H\right) < 0$, the term $\left[(1-p)\left(\rho_1 - \rho_H\right) + \mu_4\left(\rho_0 - \rho_H\right)\right]\lambda_H$ is non-negative. Since $\lambda_L = 1$ and since by assumption $p\left(\rho_1 - \rho_L\right) > 1$, the term $\left[p\left(\rho_1 - \rho_L\right) + \mu_5\left(\rho_0 - \rho_L\right)\right]\lambda_L > 1$. Therefore we have $\mu_3 > 1$ as long as $I > 0$, which implies $\mu_4 > 0$ and $\mu_5 > 0$ and establishes the claim. □

Proof of Proposition 2. We first establish that the constraints

$$I \leq \frac{A - (q-1)\,\ell}{1 - p\left(\rho_0 - \rho_L\right) - (1-p)\lambda_H\left(\rho_0 - \rho_H\right)}$$

and $\lambda_H\left(\rho_H - \rho_0\right) I \leq \ell$ hold with equality. Consider the Lagrangian

$$L = p\left(\rho_1 - \rho_L\right) I + (1-p)\lambda_H\left(\rho_1 - \rho_H\right) I + \mu_1\left[\ell - \lambda_H\left(\rho_H - \rho_0\right) I\right]$$
$$+ \mu_2\left[A - (q-1)\,\ell - \left[1 - p\left(\rho_0 - \rho_L\right) - (1-p)\lambda_H\left(\rho_0 - \rho_H\right)\right] I\right]$$

The first-order condition with respect to ℓ yields

$$\mu_1 \leq (q-1)\,\mu_2$$

which holds with equality when $\ell > 0$. Since we have $\ell \geq \lambda_H\left(\rho_H - \rho_0\right) I$, as long as we have $\lambda_H = 1$ and $I > 0$, we will have $\ell > 0$. We proved in Lemma 1 that (2.11) binds at the solution, so we have $\mu_2 > 0$, which implies that $\mu_1 > 0$ as long as we have $q - 1 > 0$, $\lambda_H = 1$, and $I > 0$. If we have $q - 1 = 0$, then there is no cost of holding unnecessary ℓ, and so the optimal level of ℓ is undetermined. We assume without loss of generality that $\ell = \lambda_H (\rho_H - \rho_0) I$ holds with equality.

Now we substitute this expression for ℓ into the leverage constraint, and derive an expression for I

$$I = \frac{A}{1 - p(\rho_0 - \rho_L) - \lambda_H (q - p)(\rho_0 - \rho_H)}$$

which will hold for $\lambda_H = 1$. This same expression will also hold for $\lambda_H = 0$ and $\ell = 0$ from the leverage constraint with $\ell = 0$, so this will hold in either case.

All that is left is to determine when $\lambda_H = 1$ will be optimal. This will be true as long as

$$I_{\lambda_H=1} \geq p I_{\lambda_H=0}$$

which simplifies to

$$q - 1 \leq \frac{(1-p)[1 - p(\rho_H - \rho_L)]}{p(\rho_H - \rho_0)}$$

□

Proof of Proposition 3. (i) follows directly from Proposition 2.

For (ii), we simply look at the demand for outside assets ℓ at $q = 1$. If $\ell < \bar{\ell}$, then we know from market clearing that equilibrium $q = 0$ because households must hold some of $\bar{\ell}$ in equilibrium. Substituting $q = 1$ into the expression for I in Proposition 2, we find that $I = \frac{A}{1-\rho_0+p\rho_L+(1-p)\rho_H}$. Since firms need to hold at least $\ell = (\rho_H - \rho_0) I$ in order to meet both shocks, we find that firm must hold at least $\ell = \frac{(\rho_H-\rho_0)A}{1-\rho_0+p\rho_L+(1-p)\rho_H}$, which is feasible if $\bar{\ell} \geq \frac{(\rho_H-\rho_0)A}{1-\rho_0+p\rho_L+(1-p)\rho_H}$, thus proving the statement.

For (iii), we want to find the range of $\bar{\ell}$ for which all firms meet both shocks and $q > 0$. Since $q > 0$, households do not hold any liquid assets, and so all $\bar{\ell}$ are held by firms. Since all firms meet both shocks and hold liquid assets, we have $\ell = \bar{\ell}$ by market clearing. Then we use $I = \frac{A}{1-p(\rho_0-\rho_L)-(q-p)(\rho_0-\rho_H)}$ and $I = (\rho_H - \rho_0)\,\ell$ from Proposition 2 to solve for implied q. This calculation yields

$$q - 1 = \frac{A}{\bar{\ell}} - \frac{1 - \rho_0 + p\rho_L + (1-p)\rho_H}{\rho_H - \rho_0}$$

This will be the equilibrium as long as firms are willing to meet both shocks at this level of q. By Proposition 2, firms are willing meet any shocks as long as

$$q - 1 \leq \frac{(1-p)[1 - p(\rho_H - \rho_L)]}{p(\rho_H - \rho_0)}$$

Combined with the condition above, this yields the necessary level of $\bar{\ell}$ for this to be an equilibrium

$$\bar{\ell} \geq \frac{p(\rho_H - \rho_0)A}{1 - p(\rho_0 - \rho_L)}$$

If $\bar{\ell}$ is above this threshold, then $\ell = \bar{\ell}$, $I = \frac{\bar{\ell}}{\rho_H - \rho_0}$, and $\lambda_1 = 1$.

For (iv), if $\bar{\ell}$ is below this threshold then there is insufficient liquidity for all firms to meet all shocks. However, no firms meeting both shocks would not be an equilibrium, because in this case the available outside liquidity $\bar{\ell}$ would need to be held by households or firms that do not need it, which they would only do if we had $q = 0$. But if we have $q = 0$, then firms would prefer to buy outside liquidity at this price and meet both shocks. Therefore in equilibrium we must have a fraction of firms meeting both shocks, while the rest only meet the low shock. In order for this to be an equilibrium the firms must be indifferent between these two strategies. From Proposition 2, this will be true if

$$q - 1 = \frac{(1-p)\left[1 - p(\rho_H - \rho_L)\right]}{p(\rho_H - \rho_0)}$$

Substituting this q into the expression for I from Proposition 2, we obtain

$$I = \frac{pA}{1 - p(\rho_0 - \rho_L)}$$

The corresponding amount of outside liquidity held by each firm must satisfy $\ell = (\rho_H - \rho_0) I$, so we have

$$\ell = \frac{pA(\rho_H - \rho_0)}{1 - p(\rho_0 - \rho_L)}$$

Let the fraction of firms that meet both shocks be ζ. Since $q > 0$, by market clearing the firms that meet both shocks must hold total liquidity $\bar{\ell}$. Therefore we have $\zeta\ell = \bar{\ell}$, and so $\zeta = \frac{\bar{\ell}}{\ell}$, or

$$\zeta = \frac{\left[1 - p(\rho_0 - \rho_L)\right]\bar{\ell}}{pA(\rho_H - \rho_0)}$$

The firms that meet only the low shock will follow the same strategy as when $p(\rho_H - \rho_L) > 1$, earning the same profits as the firms that meet both shocks. This proves the final statement. □

B Online Appendix: Proofs from section 3

Proof of Lemma 2. Let the Lagrangian for the optimal contracting problem be

$$
\begin{aligned}
L &= p\left(\lambda_L\rho_1 I - R_L^I - R_L^1 - \lambda_B R_L^B + \ell\right) + (1-p)\left(\lambda_H\rho_1 I - R_H^I - R_H^1 - \lambda_B R_H^B + \ell\right) \\
&\quad + \mu_1\left[R_L^1 - \lambda_L\rho_L I\right] + \mu_2\left[R_H^1 + \lambda_B M - \lambda_H\rho_H I\right] \\
&\quad + \mu_3\left[pR_L^I + (1-p)R_H^I - I - q\ell + A\right] \\
&\quad + \mu_4\left[\lambda_L\rho_0 I + \ell - R_L^I - R_L^1 - \lambda_B R_L^B\right] + \mu_5\left[\lambda_H\rho_0 I + \ell - R_H^I - R_H^1 - \lambda_B R_H^B\right] \\
&\quad + \mu_6\left[pR_L^B + (1-p)R_H^B - \lambda_B(\pi + (1-p)M)\right]
\end{aligned}
$$

The first step is to show that (3.3) and (3.4) bind, meaning period 1 investors are paid just enough to finance the liquidity shock. Differentiating with respect to R_L^1 and R_H^1 yields

$$
\begin{aligned}
\mu_1 &\le p + \mu_4 \\
\mu_2 &\le (1-p) + \mu_5
\end{aligned}
$$

which hold with equality if $R_L^1 > 0$ or $R_H^1 > 0$ respectively.

From (3.3) we have $R_L^1 \ge \lambda_L\rho_L I$, and so either $R_L^1 > 0$, which implies $\mu_1 = p + \mu_4 > 0$, or $R_L^1 = 0$ in which case $\lambda_L\rho_L I = 0$ (since $\lambda_L\rho_L I \ge 0$). In either case, the constraint (3.3) holds with equality.

From (3.4) we have $R_H^1 + \lambda_B M \ge \lambda_H\rho_H I$. Now we would like to argue as above, but first we must rule out the possibility that we have $R_H^1 = 0$ and $\lambda_B M > 0$, so that (3.4) does not hold with equality. To show that this is not the case, we differentiate the Lagrangian with respect to M, which yields

$$\mu_2\lambda_B \le \mu_6(1-p)M$$

which holds with equality if $M > 0$. From this we can conclude that either (1) $M = 0$, or (2) $\mu_2 > 0$ and $\mu_6 > 0$, or (3) $\mu_2 = 0$ and $\mu_6 = 0$. If (1) or (2) then we are finished then (3.4) holds with equality and we are finished, so all that remains is to show that we cannot have $\mu_2 = \mu_6 = 0$ when $\lambda_B = 1$ and $M > 0$.

Differentiating the Lagrangian with respect to R_L^B and R_H^B, we obtain

$$\mu_6 \le 1 + \frac{\mu_4}{p}$$
$$\mu_6 \le 1 + \frac{\mu_5}{1-p}$$

which hold with equality if $R_L^B > 0$ or $R_H^B > 0$ respectively. From this we conclude that either $\mu_6 > 0$ or $R_L^B = R_H^B = 0$. If the latter, then (3.2) would be $\pi + (1-p)M = 0$, which is only possible if $\pi = M = 0$, contradicting our assumption that $M > 0$. Therefore we have $\mu_6 > 0$, and so $\mu_2 > 0$.

Note that the above argument also establishes that (3.2) holds with equality, since either $R_L^B = R_H^B = 0$ and so $\lambda_B(\pi + (1-p)M) = 0$, or else one of R_L^B or R_H^B is strictly positive, in which case $\mu_6 > 0$.

We have now established that (3.3) and (3.4) hold with equality, and so we can directly substitute them into the Lagrangian to obtain

$$\begin{aligned}
L = {} & p\left(\lambda_L\rho_1 I - R_L^I - \lambda_L\rho_L I - \lambda_B R_L^B + \ell\right) + (1-p)\left(\lambda_H\rho_1 I - R_H^I - \lambda_H\rho_H I + \lambda_B(M - R_H^B) + \ell\right) \\
& + \mu_3\left[pR_L^I + (1-p)R_H^I - I - q\ell + A\right] + \mu_4\left[\lambda_L\rho_0 I + \ell - R_L^I - \lambda_L\rho_L I - \lambda_B R_L^B\right] \\
& + \mu_5\left[\lambda_H\rho_0 I + \ell - R_H^I - \lambda_H\rho_H I + \lambda_B(M - R_H^B)\right] + \mu_6\left[pR_L^B + (1-p)R_H^B - \lambda_B(\pi + (1-p)M)\right]
\end{aligned}$$

We next establish that $\lambda_L = 1$. Since $\lambda_L, \lambda_H \in \{0,1\}$, we can differentiate the Lagrangian with respect to each λ, and conclude that if this derivative is non-negative then that $\lambda = 1$. The derivatives of the Lagrangian with respect to λ_L and λ_H are respectively

$$\left[p\left(\rho_1 - \rho_L\right) + \mu_4\left(\rho_0 - \rho_L\right)\right] I$$
$$\left[(1-p)\left(\rho_1 - \rho_H\right) + \mu_5\left(\rho_0 - \rho_H\right)\right] I$$

Since by assumption $\rho_1 > \rho_L$ and $\rho_0 > \rho_L$, the first expression is positive for $I > 0$, and so we have $\lambda_L = 1$. The second expression may not be positive because $\rho_0 < \rho_H$, so we conclude that $\lambda_H = 1$ if and only if $\left[(1-p)\left(\rho_1 - \rho_H\right) + \mu_5\left(\rho_0 - \rho_H\right)\right] I \ge 0$.

Differentiating the Lagrangian with respect to R_L^I and R_H^I yields

$$\mu_4 \ge p\left(\mu_3 - 1\right)$$
$$\mu_5 \ge (1-p)\left(\mu_3 - 1\right)$$

which hold with equality when $R_L^I > 0$ or $R_H^I > 0$ respectively. Therefore (3.5) and (3.6) hold with equality as long as $\mu_3 > 1$.

Now to finish proving the proposition, it is enough to establish that $\mu_3 > 1$, since this would prove that (3.5), (3.6) and (3.1) hold with equality. Differentiating the Lagrangian with respect to I, we obtain

$$\mu_3 \geq p\left(\rho_1 - \rho_L\right) + \mu_4\left(\rho_0 - \rho_L\right) + \lambda_H\left[\left(1-p\right)\left(\rho_1 - \rho_H\right) + \mu_5\left(\rho_0 - \rho_H\right)\right]$$

By assumption, we have $p\left(\rho_1 - \rho_L\right) > 1$, so we have

$$\mu_3 > 1 + \mu_4\left(\rho_0 - \rho_L\right) + \lambda_H\left[\left(1-p\right)\left(\rho_1 - \rho_H\right) + \mu_5\left(\rho_0 - \rho_H\right)\right]$$

Since $\mu_4 \geq 0$ and $\rho_0 > \rho_L$, the term $\mu_4\left(\rho_0 - \rho_L\right) \geq 0$. As we found above, either $\lambda_H = 0$ or $\left(1-p\right)\left(\rho_1 - \rho_H\right) + \mu_5\left(\rho_0 - \rho_H\right) \geq 0$, so the term $\lambda_H\left[\left(1-p\right)\left(\rho_1 - \rho_H\right) + \mu_5\left(\rho_0 - \rho_H\right)\right] \geq 0$. Therefore we have $\mu_3 > 1$, and therefore $\mu_4 > 0$ and $\mu_5 > 0$. This proves that (3.5), (3.6) and (3.1) hold with equality. □

Proof of Proposition 4. I solve this problem in two stages. In the first stage, I suppose that $\lambda_H = 1$ and solve for the optimal way to finance investment. In the second stage I check whether it is in fact optimal to meet both shocks.

Assuming $\lambda_H = 1$, the problem of an individual firm given (M, π, q) is

$$\begin{aligned}
\max_{I,\ell,\lambda_B} \quad & \{I\} \\
\text{s.t.} \quad & I \leq \frac{A - (q-1)\,\ell - \lambda_B \pi}{\chi_1} \\
& I \leq \frac{\ell + \lambda_B M}{\rho_H - \rho_0}
\end{aligned}$$

where $I, \ell \geq 0$ and $\lambda_B \in \{0, 1\}$. The lagrangian of this problem is

$$L = I + \lambda\left(\frac{A - (q-1)\,\ell - \lambda_B \pi}{\chi_1} - I\right) + \mu\left(\frac{\ell + \lambda_B M}{\rho_H - \rho_0} - I\right)$$

The first-order condition with respect to I is

$$\lambda + \mu \geq 1$$

which holds with equality if $I > 0$. This implies that one of λ or μ is greater than 0.

The first-order condition with respect to ℓ is

$$\frac{\mu}{\rho_H - \rho_0} \leq \frac{\lambda(q-1)}{\chi_1}$$

which holds with equality if $\ell > 0$. Since one of λ or μ is strictly positive, this implies that $\lambda > 0$. Moreover, $\mu = 0$ is only possible if either $\ell = 0$ or $q = 1$.

First suppose $q = 1$. Then since $\lambda > 0$, we have

$$I = \frac{A - \lambda_B \pi}{\chi_1}$$

and since $\pi \geq 0$, this term attains its maximum at $\lambda_B = 0$. This level of investment is feasible, and corresponds to $\ell = (\rho_H - \rho_0)I$.

Next suppose that $q > 1$. Then $\mu > 0$ unless $\ell = 0$, which only occurs when firms meet all liquidity shocks through bank financing. Differentiating the Lagrangian with respect to λ_B yields

$$\frac{\partial L}{\partial \lambda_B} = \frac{\mu M}{\rho_H - \rho_0} - \frac{\lambda \pi}{\chi_1}$$

Firms will only choose $\lambda_B = 1$ if $\partial L / \partial \lambda_B \geq 0$. But since $\lambda > 0$, this is only possible if either $\mu > 0$ or $\pi = 0$.

First suppose that $\pi = 0$. If $\ell = 0$ also, then investment satisfies $I = A/\chi_1$, and the liquidity constraint implies $M \geq (\rho_H - \rho_0)A/\chi_1$. Now we assumed that $M \leq (\rho_H - \rho_0)A/\chi_1$, so then either M is exactly at this level, in which case this is the solution, or else we have arrived at a contradiction and $\ell > 0$.

In the latter case, $\ell > 0$ implies $\mu > 0$, and we have

$$\frac{\mu}{\rho_H - \rho_0} = \frac{\lambda(q-1)}{\chi_1}$$

Substituting this into the FOC wrt λ_H yields

$$\frac{\partial L}{\partial \lambda_B} = \frac{\lambda}{\chi_1}\left[(q-1)M - \pi\right]$$

which is positive when $\pi / M \leq q - 1$. Moreover, $\mu > 0$ implies that

$$(\rho_H - \rho_0)I = \ell + \lambda_B M$$

Combining this with the leverage constraint arising from pledgeability, we obtain

$$I = \frac{A - \lambda_B \left[\pi - (q-1)M\right]}{\chi_1 + (q-1)(\rho_H - \rho_0)}$$

Further observe that this expression still holds for investment in the $q = 1$ and $\ell = 0$ cases considered above.

Now it only remains to determine whether it is optimal to meet the high shock. When the high shock is not met, firm expected profits are

$$\frac{pA}{1 - p(\rho_0 - \rho_L)}$$

and when both shocks are met expected profits are I as calculated above. The expression given in the proposition is a comparison of these values. □

Proof of Proposition 5. First consider the case $\tilde{D}(q-1) \leq \bar{D}(q-1)$. We begin by arguing that $\tilde{D}(\vartheta)$ is increasing in ϑ. Recall that $\tilde{D}(\vartheta)$ is implicitly defined by

$$\vartheta \tilde{D} = C(\tilde{D}, K)$$

Implicitly differentiating this expression yields

$$\tilde{D}'(\vartheta) = \frac{\tilde{D}}{C_1(\tilde{D}, K) - C(\tilde{D}, K)/\tilde{D}}$$

which is positive by our assumptions $C(0, K) = 0$, $C_1 > 0$, and $C_{11} > 0$.

Now if any bank sets $\vartheta > q - 1$, it sells no credit lines, whereas any bank that sets $\vartheta \leq q - 1$ will sell some quantity of credit lines that cannot exceed $\tilde{D}(\vartheta)$, since this is the quantity of D at which its agency costs are binding. Since $\tilde{D}(\vartheta)$ is increasing in ϑ, total credit lines sold by all firms cannot exceed $\tilde{D}(q-1)$. Since this quantity is by assumption no greater than total demand for credit lines $\bar{D}(\vartheta)$, any bank that sets $\vartheta \leq q - 1$ will sell up to its agency costs $\tilde{D}(\vartheta)$. Thus profits are $\vartheta \tilde{D}(\vartheta)$ for $\vartheta \leq q - 1$, and 0 for $\vartheta > q - 1$, and since $\tilde{D}(\vartheta)$ is increasing in ϑ, the solution is $\vartheta = q - 1$.

Next consider the case $\tilde{D}(q-1) < \bar{D}(q-1)$. $\bar{D}(\vartheta) = \frac{(\rho_H - \rho_0)A}{\chi_1 + \vartheta(\rho_H - \rho_0)}$ is the maximum demand for credit lines from firms given a price of liquidity equal to ϑ. Clearly $\bar{D}(\vartheta)$ is decreasing in ϑ.

Given some distribution $\Phi(\cdot)$ of ϑ across banks, with $\Phi(q-1) = 1$, firms will purchase credit lines from the banks with the lowest ϑ until agency costs bind for those banks, and then move on to the next highest, and so on. Thus for a given ϑ, the total share of firms that have purchased credit lines from banks at a price less than or equal to this price will be

$$S(\vartheta) = \int_0^{\vartheta} \frac{\bar{D}(\vartheta')}{\tilde{D}(\vartheta')} d\Phi(\vartheta')$$

Firms purchase credit lines from banks up to a cutoff price $\bar{\vartheta}$. Since at ϑ the share of firms that have purchased credit lines is 1, this cutoff is defined by

$$\lim_{\vartheta \to \bar{\vartheta}^-} S(\vartheta) \leq 1 \leq \lim_{\vartheta \to \bar{\vartheta}^+} S(\vartheta)$$

where both inequalities hold with equality if there is not a point mass of firms at $\bar{\vartheta}$.

If there is a point mass $\bar{m}$ of banks at $\bar{\vartheta}$, then demand will be divided equally among them. Each will sell credit lines to a share $\phi = [1 - \lim_{\vartheta \to \bar{\vartheta}^-} S(\vartheta)]/\bar{m}$, of firms, and each firm purchases a credit line of size $\bar{D}(\bar{\vartheta})$.

Since $\bar{D}(\cdot)$ is decreasing in ϑ whereas $\tilde{D}(\cdot)$ is increasing, the term $\frac{\bar{D}(\vartheta)}{\tilde{D}(\vartheta)}$ is decreasing in ϑ. Moreover, because we are considering the case $\bar{D}(q-1) > \tilde{D}(q-1)$, this fraction is strictly greater than 1 for all $\vartheta \leq 1$. Therefore any bank that sets $\vartheta = q - 1$ will receive a measure of customers that is strictly less than 1 (and may be 0).

Now consider the problem of a bank setting its price. Clearly the bank will never choose $\vartheta > \bar{\vartheta}$, since this yields no sales and no profits. If the bank chooses to just undersell the cutoff, it receives profits of just under $\bar{\vartheta}\tilde{D}(\bar{\vartheta})$. Finally, if the bank chooses $\vartheta = \bar{\vartheta}$, it receives profits $\phi\bar{\vartheta}\bar{D}(\bar{\vartheta})$.

Since all banks face the same demand curve which implies a unique optimal price, they will all choose the same ϑ. This point must satisfy $\tilde{D}(\bar{\vartheta}) = \phi\bar{D}(\bar{\vartheta})$. Since all banks choose the same price, $\phi = 1$ and this expression becomes $\tilde{D}(\bar{\vartheta}) = \bar{D}(\bar{\vartheta})$, which implicitly defines the price level set by banks. □

Proof of Proposition 6. Equilibrium is defined by firm behavior given by Proposition 4, bank pricing given by Proposition 5, and the market clearing condition $(\zeta\ell - \bar{\ell})(q-1) = 0$, together with $\zeta\ell \leq \bar{\ell}$ and $q \geq 1$. Market clearing requires that when $q > 1$, firms must hold all available outside liquidity $\bar{\ell}$, since households will only hold ℓ if $q = 1$.

First suppose that $q = 1$. This immediately implies $\zeta = 1$. From bank behavior, we know that $C(M, K)/M \leq q - 1 = 0$, which is only possible when $M = 0$. Then from firm behavior we have $I = I_1(0) = A/\chi_1$. By assumption, $I_1(0) > pI_0$, so $\lambda_H = 1$ for all firms. Therefore firms meet all shocks by holding outside liquidity, and so total holdings must satisfy $\ell = (\rho_H - \rho_0) I_1(0)$. This will be the equilibrium as long as there is sufficient outside liquidity, i.e. $\bar{\ell} \geq (\rho_H - \rho_0) I_1(0)$.

Next suppose that $\bar{\ell} < (\rho_H - \rho_0) I_1(0)$, so that $q > 1$ in equilibrium. This implies that all outside liquidity is held by firms, and so $\zeta \ell = \bar{\ell}$. Now I claim that banks set $\vartheta = q - 1$. First suppose that $\bar{\ell} > 0$. From Proposition 5, either $\vartheta = q - 1$, or else $\vartheta < q - 1$ and $\bar{M}(\vartheta) = \tilde{M}(\vartheta)$. In the latter case, all firms finance their desired liquidity holdings by holding credit lines, and so $\ell = 0$. Since $\bar{\ell} > 0$, we have $\zeta \ell < \bar{\ell}$, which violates market clearing given $q > 1$. Now if $\bar{\ell} = 0$, q may be set to any level as long as $\vartheta \leq q - 1$, so we can set them equal without loss of generality.

Since $\vartheta = q - 1$, a firm that meets a high liquidity shock will choose investment

$$I = I_1(q-1) = \frac{A}{\chi_1 + (q-1)(\rho_H - \rho_0)}$$

and will be indifferent between purchasing a credit line and financing liquidity shocks entirely through holding outside liquidity. We assume without loss of generality that each firm that meets the high shock holds a credit line of size M. Then $\tilde{M} = \zeta M$ are total credit lines sold by banks. From Proposition 5, $\tilde{M}$ satisfies $\vartheta \tilde{M} = C(\tilde{M}, K)$. Combined with $\vartheta = q - 1$, this implies $q - 1 = C(\tilde{M}, K)/\tilde{M}$.

Now we consider two cases. First suppose that $\zeta = 1$, so that all firms meet the high shock. Then $\tilde{M} = M$ and $\ell = \bar{\ell}$. Then from firm behavior

$$(\rho_H - \rho_0) I_1(C(M, K)/M) = M + \bar{\ell}$$

This expression implicitly defines unique $M = M_1$. To prove this, frist note that by L'Hopital's rule, $\lim_{M \to 0} C(M, K)/M = 0$, since $C(0, K) = 0$ and $\lim_{M \to 0} C_1(M, K) = 0$. Then as $M \to 0$, the right-hand side converges to ℓ while the left-hand side converges to $(\rho_H - \rho_0) I_1(0)$. Since we are considering the case that $(\rho_H - \rho_0) I_1(0) > \bar{\ell}$, the left-hand side is greater than the right at $M \to 0$. Next we observe that since $I_1(q-1)$ is strictly decreasing in $q - 1$, and since $C(M, K)/M$ is strictly increasing in M, the left-hand side is strictly decreasing in M whereas the right-hand side is strictly decreasing. Therefore if an M that makes the equation holds, it is unique. Finally, by assumption $\lim_{M \to \infty} C(M, K) = \infty$ and $\lim_{M \to \infty} C_1(M, K) = \infty$, so again by L'Hopital's rule $\lim_{M \to \infty} C(M, K)/M = \infty$. Since $\lim_{q \to \infty} I_1(q-1) = 0$, this implies that as $M \to \infty$ the left-hand side converges to 0 while the right-hand side converges to ∞. Thus M_1 exists.

The equilibrium will satisfy $M = M_1$ as long as the implied I is at least as large as pI_0.

This condition can be written as

$$I_1(C(M_1, K)/M_1) \geq pI_0$$

Since $C(M,K)/M$ is strictly increasing in M, and $I_1(q-1)$ is strictly decreasing in $q-1$, this expression is equivalent to the condition $M_1 \leq M_2$, where M_2 satisfies $I_1(C(M_2,K)/M_2) = pI_0$. Note that M_2 is uniquely defined since by the arguments above $I_1(C(M,K)/M)$ is strictly decreasing in M, goes to 0 as $M \to \infty$, and goes to $I_1(0) > pI_0$ as $M \to 0$.

Suppose that instead $M_2 < M_1$. Then we have $\zeta < 1$. Moreover, since $\tilde{M}$ satisfies $C(\tilde{M},K)/\tilde{M} = q-1 > 0$, we have $\tilde{M} > 0$, and since $\tilde{M} = \zeta M$, we must have $\zeta > 0$. Thus $\zeta \in (0,1)$, and so firms must be indifferent between meeting the high shock or the low shock. Thus firms that meet the high shock choose $I = pI_0$. This implies that $q-1$ must satisfy

$$I_1(q-1) = pI_0$$

which uniquely defines a value of $q > 1$. Then $\tilde{M}$ is determined by $q-1 = C(\tilde{M},K)/\tilde{M}$, which is what we're calling M_2. Now we know that total liquidity used satisfies $\tilde{L} = M_2 + \bar{\ell}$, and liquidity used by each firm that meets a high shock is $L = M + \ell = (\rho_H - \rho_0)I$. Therefore the fraction of firms that meet the high shock is

$$\zeta = \frac{M_2 + \bar{\ell}}{(\rho_H - \rho_0)I}$$

□

C Online Appendix: Proofs from section 4

Proof of Proposition 7. At an interior equilibrium the supply of liquidity is

$$L^s(q-1) = M^s(q-1) + \bar{\ell}$$

where $M^s(q-1)$ is implicitly defined by $C(M,K)/M = q-1$, and demand for liquidity is

$$L^d(q-1) = \frac{(\rho_H - \rho_0)A}{\chi_1 + (q-1)(\rho_H - \rho_0)}$$

Equilibrium q is defined implicitly by $L^s(q-1) = L^d(q-1)$, and at the equilibrium we have $q-1 = C(M,K)/M$ and $L^s = L^d = M + \bar{\ell}$.

The elasticity of liquidity supply at equilibrium is

$$\eta = \left(\frac{C/M}{C_M - C/M}\right)\left(\frac{M}{M+\bar{\ell}}\right)$$

and the elasticity of liquidity demand is

$$\epsilon = \left(M+\bar{\ell}\right)\frac{C/M}{A}$$

Now consider a marginal change in $\bar{\ell}$. Totally differentiating $L^s = L^d$ yields $L^s_q dq + d\bar{\ell} = L^d_q dq$, which implies

$$\frac{dq}{d\bar{\ell}} = -\left(\frac{C_M - C/M}{M}\right)\left(\frac{\eta}{\eta+\epsilon}\right)$$

The total change in $L = M + \bar{\ell}$ is $L^s_q dq$, since the demand curve is not affected directly by the change in $\bar{\ell}$. This implies $1 + \frac{dM}{d\bar{\ell}} = L^d_q \frac{dq}{d\bar{\ell}}$, or

$$\frac{dM}{d\bar{\ell}} = -\left(\frac{\eta}{\eta+\epsilon}\right)$$

Differentiating the expression for investment $I = (\rho_H - \rho_0)^{-1}\left(M+\bar{\ell}\right)$ yields $\frac{dI}{d\bar{\ell}} = (\rho_H - \rho_0)^{-1}\left(1 + \frac{dM}{d\bar{\ell}}\right)$, or

$$\frac{dI}{d\bar{\ell}} = (\rho_H - \rho_0)^{-1}\left(\frac{\epsilon}{\eta+\epsilon}\right)$$

□

Proof of Proposition 8. Equilibrium satisfies $L^s(q-1) = L^d(q-1)$, where L^s and L^d are as defined in the proof of Proposition 7. Then a marginal change in K produces a change in q equal to

$$\frac{dq}{dK} = \frac{L^d_K}{L^s_q - L^d_q}$$

We can calculate L^d_K by applying the implicit function theorem to $C(M,K) = (q-1)M$. From this we find $L^d_K = -\frac{C_K}{M}L^d_q$, which simplifies to

$$\frac{dq}{dK} = \left(\frac{\eta}{\eta+\epsilon}\right)\frac{C_K}{M}$$

Since K does not appear in the definition of L^d, we have $\frac{dM}{dK} = D_q \frac{dq}{dK}$, or

$$\frac{dM}{dK} = -\left(\frac{M}{C_M - C/M}\right)\left(\frac{\epsilon}{\epsilon+\eta}\right)\frac{C_K}{M}$$

and since $I = (\rho_H - \rho_0)^{-1}(M + \bar{\ell})$, we have

$$\frac{dI}{dK} = (\rho_H - \rho_0)^{-1}\frac{dM}{dK}$$

□

D Online Appendix: Proofs from section 5

Proof of Proposition 9. Since firms prefer the unconstrained optimum (UO) to any other allocation, it is sufficient to show that UO is feasible under the described set of transfers. The UO requires firms to invest $I = A + H_0$. Firms have an endowment of A, and so given the transfer H_0 from households, this level of investment is feasible. The UO also requires meeting both shocks. Since firms have no debt from period 0, they have sufficient pledgeable funds to borrow up to $\rho_0 I$ from households in period 1. Since $\rho_0 > \rho_L$, firms that experience the low shock are able to borrow enough to meet the shock. Firms that receive the high shock receive a transfer of $(\rho_H - \rho_0)I$ from households. Since these firms can borrow an additional $\rho_0 I$ from households, they are able to meet the high shock. Thus the unconstrained optimum is feasible under the described transfers. The Planner may then transfer $H_0 + (1-p)(\rho_H - \rho_0)(A + H_0)$ from firms to households in period 2 to effect a Pareto improvement. □

Proof of Proposition 10. We solve for the constrained optimal allocation in two steps. First we fix the fraction ζ of firms that meet the high shock and define welfare $W(\zeta)$. Then we solve for ζ that maximizes $W(\zeta)$.

First consider the optimal choices of firms that choose $\lambda_H = 0$. This choice reduces to maximizing $\int_i p(\rho_1 - \rho_L)I_i$ s.t. $I_i \leq A/\left[1 - p(\rho_0 - \rho_L)\right]$. This yields optimal investment $I_i = A/\left[1 - p(\rho_0 - \rho_L)\right]$, and their welfare is

$$W_{\lambda=0} = \frac{R_0 A}{\chi_0}$$

where $R_0 = p(\rho_1 - \rho_L) - 1$ and $\chi_0 = 1 - p(\rho_0 - \rho_L)$.

Next consider the problem of firms that choose $\lambda_H = 1$. The Lagrangian of the maximization problem in Definition 2 with ζ given is

$$
\begin{aligned}
L = \int_i R_1 I_i + \int_i \mu_1^i \left(\frac{A - \pi_i - (q-1)\ell_i}{\chi_1} - I_i \right) \\
+ \int_i \mu_2^i \left(M_i + \ell_i - I_i(\rho_H - \rho_0) \right) + \mu_3 \left(\bar{\ell} - \int_i \ell_i \right) \\
+ \mu_4 \left(\int_i \pi_i - C(\int_i M_i, K) \right) + \mu_5 (q-1)
\end{aligned}
$$

together with non-negativity constraints. The first-order conditions of the problem are:

$$
\begin{aligned}
\frac{\partial L}{\partial I_i} &= R_1 - \mu_1^i - \mu_2^i(\rho_H - \rho_0) \leq 0 \\
\frac{\partial L}{\partial M_i} &= \mu_2^i - C_1 \mu_4 \leq 0 \\
\frac{\partial L}{\partial \pi_i} &= \mu_4 - \frac{\mu_1^i}{\chi_1} \leq 0 \\
\frac{\partial L}{\partial \ell_i} &= \mu_2^i - \frac{\mu_1^i (q-1)}{\chi_1} - \mu_3 \leq 0 \\
\frac{\partial L}{\partial q} &= \mu_5 - \frac{\mu_1^i \ell_i}{\chi_1} \leq 0
\end{aligned}
$$

Now I establish several facts about the solution.

First I claim that $\mu_1^i > 0$ for all i. I argue in two steps. First I show that at least one of $\mu_1^i, \mu_2^i > 0$. This follows from the expression for $\partial L / \partial I_i$ since $R_1 > 0$, $\rho_H > \rho_0$, and $\mu_1^i, \mu_2^i \geq 0$. Next I argue that $\mu_1^i > 0$. If this were not true, then by the previous claim $\mu_2^i > 0$. Then from the expression for $\partial L / \partial M_i$ it follows that $\mu_4 > 0$, and from the expression for $\partial L / \partial \pi_i$ it follows that $\mu_1^i > 0$, which is a contradiction.

Next I show that $q = 1$. From the expression for $\partial L / \partial q$ either $\ell_i = 0$ for all i, or else $\mu_5 > 0$. The latter implies $q = 1$, and if the former the value of q does not enter into the problem apart from the restriction that $q \geq 1$, and so we can choose $q = 1$ without loss of generality.

Next I argue that $\int_i \pi_i = C(\int_i M_i, K)$. This follows from the expression for $\partial L_i / \partial \pi_i$. If $\pi_i > 0$ for any i then $\mu_4 > 0$ and the claim follows. If $\pi_i = 0$ for all i, then the bank incentive compatability constraint becomes $C(\int_i M_i, K) \leq 0$, and since we have $C(\int_i M_i, K) \geq 0$ from non-negativity of M_i, the claim follows. Moreover, this argument together with the expression for $\partial L / \partial M_i$ implies that either $\mu_2^i > 0$, or $M_i, \pi_i = 0$ for all i.

Now observe that since total welfare satisfies $\int_i \left(\frac{A - \pi_i}{\chi_1} \right) R_1$, any feasible reallocation of π_i, M_i, ℓ_i across i will not affect total welfare. Thus we can WLOG restrict attention to the symmetric case where $M_i = M$, $\pi_i = \pi$, and $\ell_i = \ell$.

Combining all of the above, we can rewrite the maximization problem as

$$\begin{aligned} &\max \left\{ \frac{A - C(\zeta M, K)/\zeta}{\chi_1} \right\} \\ &\text{s.t.} \quad \frac{A - C(\zeta M, K)/\zeta}{\chi_1} \leq \frac{M + \ell}{\rho_H - \rho_0} \\ &\qquad \zeta \ell \leq \bar{\ell},\ M \geq 0 \end{aligned}$$

Clearly if $M = 0$ is feasible, the solution is $M = 0$ and $I = A/\chi_1$. This will be feasible only when $\frac{A}{\chi_1} \leq \frac{\bar{\ell}/\zeta}{\rho_H - \rho_0}$, which we can write as a threshold $\bar{\ell} \geq \zeta(\rho_H - \rho_0)\frac{A}{\chi_1}$.

Combining expressions, we obtain a single expression that implicitly defines $I_{\lambda=1}(\zeta)$:

$$\chi_1 I = A - \frac{C\left(\max(\zeta(\rho_H - \rho_0)I - \bar{\ell}, 0), K\right)}{\zeta}$$

and the welfare of firms that meet the high shock is $W_{\lambda=1}(\zeta) = R_1 I_{\lambda=1}(\zeta)$.

Now we turn to the determination of optimal ζ. We can now express total welfare as

$$W(\zeta) = \zeta W_{\lambda=1}(\zeta) + (1 - \zeta) W_{\lambda=0}$$

Marginal welfare satisfies

$$W_\zeta = R_1 I_{\lambda=1} - \frac{R_0 A}{\chi_0} + \zeta R_1 \frac{dI_{\lambda=1}}{d\zeta}$$

First observe that if ζ is small enough that $\zeta(\rho_H - \rho_0)\frac{A}{\chi_1} \leq \bar{\ell}$, $I_{\lambda=1} = A/\chi_1$ and $W_\zeta = \frac{R_1 A}{\chi_1} - \frac{R_0 A}{\chi_0}$. Now we show that this expression is strictly positive under our assumptions. Recall that we assumed it was always profitable to meet the high shock in equilibrium when the liquidity premium was 0. This assumption can be written as $1/\chi_1 > p/\chi_0$, and is equivalent to $p(\rho_H - \rho_L) < 1$. Then

$$\begin{aligned} \frac{R_1}{\chi_1} - \frac{R_0}{\chi_0} &> \frac{pR_1}{\chi_0} - \frac{R_0}{\chi_0} \\ &= \frac{1}{\chi_0}\left[p\left(\rho_1 - p\rho_L - (1-p)\rho_H - 1\right) - \left(p(\rho_1 - \rho_L) - 1\right)\right] \\ &= \frac{1-p}{\chi_0}\left[1 - p\left(\rho_H - \rho_L\right)\right] > 0 \end{aligned}$$

Thus for ζ sufficiently low, $W_\zeta > 0$.

Now we show that W_ζ is strictly decreasing in ζ for all ζ above this level. Applying the implicit function theorem to our expression for $I_{\lambda=1}(\zeta)$, we obtain

$$\frac{dI}{d\zeta} = \frac{C/\zeta - (\rho_H - \rho_0)C_1 I}{\zeta\left[(\rho_H - \rho_0)C_1 + \chi_1\right]}$$

which implies

$$W_\zeta = \frac{R_1 A}{(\rho_H - \rho_0)C_1 + \chi_1} - \frac{R_0 A}{\chi_0}$$

In the expression, ζ only appears in the argument of C_1 as ζI. As ζ increases, the term ζI increases:

$$\frac{d(\zeta I)}{d\zeta} = I + \zeta\frac{dI}{d\zeta} = \frac{\zeta\chi_1 I + C}{\zeta\left[(\rho_H - \rho_0)C_1 + \chi_1\right]} > 0$$

and as ζI increases, $C_1\left(\zeta(\rho_H - \rho_0)I - \bar{\ell}, K\right)$ decreases, since we assumed $C_{11} < 0$. Therefore $W_{\zeta\zeta} < 0$.

Since for some $\zeta > 0$ we have $W_\zeta > 0$, and since W_ζ is strictly decreasing in ζ, either $W_\zeta = 0$ for some $\zeta \leq 1$, or else $W_\zeta(1) > 0$. Thus optimal ζ is defined by

$$\frac{R_1}{(\rho_H - \rho_0)C_1\left(\zeta(\rho_H - \rho_0)I - \bar{\ell}, K\right) + \chi_1} \geq \frac{R_0}{\chi_0}$$

which will hold with equality unless $\zeta = 1$. By setting $\zeta = 1$ in this expression, we can solve for the cutoff level $\hat{\ell}$ below which $\zeta < 1$ will hold. Then this expression with equality implicitly defines optimal ζ for $\bar{\ell} < \hat{\ell}$. □

E Online Appendix: Proofs from section 6

Proof of Proposition 11. In turn,

(i) Suppose not. Then $RI - C(x)$ is maximized at $x = 0$. Let $f(x) = RI|_x$. Then since the function $f(x) - C(x)$ achieves a maximum at $x = 0$, f and C must satisfy

$$f(0) - C(0) \geq f(h) - C(h)$$

for every $h > 0$. Rearranging and dividing by h, we can write this as

$$\frac{C(h) - C(0)}{h} \geq \frac{f(h) - f(0)}{h}$$

this inequality will be preserved under taking limits

$$\lim_{h\to 0}\frac{C(h)-C(0)}{h}\geq \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

which is just the definition of the derivatives of C and f at $x=0$. Therefore we have

$$C'(0)\geq f'(0)$$

Since we have $C'(0)=0$, this implies that $f'(0)=R\left.\frac{dI}{dx}\right|_{x=0}\leq 0$. But from Proposition 7 we have $\frac{dI}{dx}>0$ for $\bar{\ell}>0$, so this is a contradiction.

(ii) Any interior solution satisfies the optimality condition

$$R\frac{dI}{dx}=D'(x)$$

which is a zero of the function

$$f(\cdot)=R\frac{dI}{dx}-D'(x)=0$$

The partial derivatives of f are

$$\begin{aligned}\frac{\partial f}{\partial x}&=R\frac{d^2I}{dx}-D''(x)\\ \frac{\partial f}{\partial\bar{\ell}}&=R\frac{d^2I}{dx}\\ \frac{\partial f}{\partial K}&=R\frac{d^2I}{dxdK}\end{aligned}$$

making use of the fact that $\frac{d^2I}{d\bar{\ell}dx}=\frac{d^2I}{dx^2}$. Since this is a unique maximum of a continuously differentiable function the implicit function theorem is valid. Therefore we have

$$\begin{aligned}\frac{dx}{d\bar{\ell}}&=-\frac{R\frac{\partial^2I}{\partial x^2}}{R\frac{\partial^2I}{\partial x^2}-D''(x)}\\ \frac{dx}{dK}&=-\frac{R\frac{\partial^2I}{\partial x\partial K}}{R\frac{\partial^2I}{\partial x^2}-D''(x)}\end{aligned}$$

(iii) If the point x^* is an interior solution, then it must also be a local maximum. At a local maximum of a global function, the function is locally concave, meaning that the second derivative is negative. Here the function we are maximizing is $RI-D(x)$. Since this function is twice continuously differentiable, it will be concave at the point x^* iff the second derivative $R\frac{\partial^2 I}{\partial x^2} - D''(x) < 0$, which immediately implies the given condition.

(iv) Since we have $R\frac{\partial^2 I}{\partial x^2} - D''(x) < 0$, the denominators in the expressions for $dx/d\bar{\ell}$ and dx/dK are negative. Therefore each expression will be positive iff the numerator is positive. Therefore we obtain the given statement. □

Proof of Proposition 12. To prove (i), we can simply observe that x^* is implicitly defined by $R\phi L_x = D_x$, and that $L_x = (1 + \eta/\epsilon)^{-1}$. Therefore a higher value of η/ϵ at every value of x implies a higher L_x at every level of x. Thus at the old x^* we have $L_x > D_x$, and so it is optimal to increase x from that point, implying a higher value of x^*. Note that any lower value of x will not be the optimum, because at the old value of η/ϵ these points had lower cumulative welfare. This implies that integrating $RI_x - D_x$ between that earlier point and the old x^* yielded a positive number, and since we have now strictly increased I_x this integral has increased, and thus must still be positive. Thus the cumulative net benefit of raising x to at least x^* must be greater than before.

For (ii), we again use $L_z = (1 + \eta/\epsilon)^{-1}$. Let $\psi = \eta/\epsilon$ be the ratio of elasticities. We want to figure out what happens to ψ when we increase z. We can write this derivative as

$$\frac{\psi_z}{\psi} = \frac{\eta_z}{\eta} - \frac{\epsilon_z}{\epsilon}$$

Now we derive expressions for both terms. Using $\eta = \eta_M\left(\frac{L-z}{L}\right)$, we derive

$$\frac{\eta_z}{\eta} = \frac{\frac{d}{dz}(\eta_M)}{\eta_M} - \left(\frac{1}{L-z}\right)\left(\frac{(1+\psi)L - z}{(1+\psi)L}\right)$$

and using the expression $\epsilon = \frac{L\vartheta}{A}$, we derive

$$\frac{\epsilon_z}{\epsilon} = -\frac{1}{L}\left(\frac{1-\epsilon}{\eta+\epsilon}\right)$$

Combining these two expressions, we obtain

$$\frac{\psi_z}{\psi} = \frac{\frac{d}{dz}(\eta_M)}{\eta_M} - \left(\frac{1}{L-z}\right)\left(\frac{(1+\psi)L - z}{(1+\psi)L}\right) + \frac{1}{L}\left(\frac{1-\epsilon}{\eta+\epsilon}\right)$$

and after a bit of work, we obtain the given expression:

$$\frac{\psi_z}{\psi} = \frac{\frac{d}{dz}(\eta_M)}{\eta_M} + \frac{1 - 2\epsilon - \eta_M}{L(\epsilon+\eta)}$$

For (iii), first observe that if bank liquidity supply is isoelastic, then η_M is constant, and so $\frac{d}{dM}\eta_M = 0$. Then since $L(\epsilon + \eta) > 0$, for $L_{zz} < 0$ we need $\eta_M + 2\epsilon > 1$. Since $\epsilon = \vartheta L / A$, and at an interior solution we have $\vartheta > 0$, it follows that $\epsilon > 0$, and so $\eta_M \geq 1$ implies that $L_{zz} < 0$.

For (iv), observe that we can write the expression for L_K as

$$L_K = -\left(\frac{\vartheta_K}{\vartheta_M}\right) L_z$$

Taking the derivative yields

$$\frac{L_{Kz}}{L_z} = \left(-\frac{\vartheta_K}{\vartheta_M}\right)\left[\frac{L_{zz}}{L_z} - \left(\frac{\vartheta_{MK}}{\vartheta_K} - \frac{\vartheta_{MM}}{\vartheta_M}\right)(1 - L_z)\right]$$

Then since $\vartheta_M > 0$ and $\vartheta_K < 0$, the term $-\frac{\vartheta_K}{\vartheta_M}$ is positive. Since $L_z > 0$, we find that $L_{Kz} > 0$ iff

$$\frac{L_{zz}}{L_z} - \left(\frac{\vartheta_{MK}}{\vartheta_K} - \frac{\vartheta_{MM}}{\vartheta_M}\right)(1 - L_z) > 0$$

which is equivalent to the given expression.

□